Prescribing geodesics and a variational problem for Riemannian metrics

THOMAS METTLER

ABSTRACT. Given a prescription of unparametrised paths on a manifold M, one path for each tangent direction, we may ask whether these paths agree with the geodesics of a Riemannian metric on M. Generically, this is not the case. Motivated by this fact, we introduce a non-negative functional $\mathcal{E}$ on the space of Riemannian metrics on M so that $\mathcal{E}(g) = 0$ if and only if the geodesics of the metric g agree with the prescribed paths. We compute the variational equations for $\mathcal{E}$ and show that the conformal variational equation is, perhaps surprisingly, of Yamabe type. This allows us to obtain existence results for conformally critical points of $\mathcal{E}$. In particular, in the surface case, every conformal class contains a conformally critical metric, unique up to homothety. As a by-product, we establish that the Blaschke metric of a properly convex projective surface is a critical point for $\mathcal{E}$.

1. Introduction

A prescription of unparametrised paths on a manifold M, one path for each tangent direction, specifies the geometric shape of a family of curves but not the speed at which one travels along them. It is natural to require these paths to be the geodesics of a torsion-free connection on TM, since the geodesics of a Riemannian metric are precisely those of its Levi-Civita connection. Such a prescription is encoded by a *projective structure*: an equivalence class $\mathfrak{p}$ of torsion-free connections on TM, where two such connections are equivalent if they have the same geodesics up to parametrisation. Here by a geodesic for $\nabla \in \mathfrak{p}$ we mean an immersed curve $\gamma : I \to M$ so that $\nabla_{\dot\gamma}\dot\gamma$ vanishes identically. A projective structure $\mathfrak{p}$ is called *metrisable* if there exists a Riemannian metric g on M whose Levi-Civita connection ${}^g\nabla$ is an element of $\mathfrak{p}$. A generic projective structure is not metrisable, and necessary and sufficient conditions for metrisability of a two-dimensional projective structure were obtained by Bryant, Dunajski and Eastwood [BDE09].

A manifold M equipped with a projective structure will be called a *projective manifold*. For a closed oriented projective manifold $(M, \mathfrak{p})$ of dimension $n \geqslant 2$, we introduce a functional $\mathcal{E}$ on the space $\mathrm{Riem}(M)$ of Riemannian metrics on M as follows. Given a representative $\nabla \in \mathfrak{p}$, the difference $\nabla - {}^g\nabla$ is a 1-form on M with values in the endomorphisms of TM; we write $(\cdot)_0$ for the part of such a 1-form taking values in the trace-free endomorphisms of TM and

Date: 18th May 2026.

define

$$\mathcal{E} : \mathrm{Riem}(M) \to \mathbb{R}, \qquad g \mapsto \mathrm{Vol}_M(g)^{(2-n)/n} \int_M |(\nabla - {}^g\nabla)_0|_g^2 d\mu_g,$$

where $d\mu_g$ denotes the volume form of g, $\mathrm{Vol}_M(g)$ the volume of M with respect to $d\mu_g$ and $|\cdot|_g$ the point-wise norm induced by g on the fibres of $S^2(T^*M) \otimes TM$. It turns out that the non-negative functional $\mathcal{E}$ depends only on the projective equivalence class of ∇ and, moreover, that $\mathcal{E}(g) = 0$ for $g \in \mathrm{Riem}(M)$ if and only if $\mathfrak{p}$ is metrisable by the metric g. The presence of the factor $\mathrm{Vol}_M(g)^{(2-n)/n}$ makes $\mathcal{E}$ invariant under homothety, that is, $\mathcal{E}(cg) = \mathcal{E}(g)$ for all positive constants c and all metrics $g \in \mathrm{Riem}(M)$.

Setting $\Phi_g := (\nabla - {}^g\nabla)_0$, the variational equations for $\mathcal{E}$ take the form (see Proposition 3.1 for details)

$$T_g(\Phi_g) - 2\ell_g^*(\Phi_g) = kg,$$

where k is a constant and $T_g(\Phi_g) \in \Gamma(S^2(T^*M))$ is a purely algebraic term in Φ_g. The map ℓ_g^* is the formal adjoint of a first-order differential operator ℓ_g obtained from the linearisation of the map $g \mapsto {}^g\nabla$, which assigns to a metric its Levi-Civita connection.

An important class of (in general) non-metrisable projective surfaces carrying a distinguished metric – the so-called *Blaschke metric* – is given by the properly convex projective surfaces. They are of particular interest due to their connection to higher Teichmüller theory. For background we refer to the landmark paper by Hitchin [Hit92]; see also [Wie18] for a recent survey. Given a Riemann surface structure on M, every holomorphic cubic differential C determines a Riemannian metric g – the Blaschke metric – via Hitchin's self-duality equations [Hit87], and the pair (g, C) encodes a properly convex projective structure on M [Lof01, Lab07]. Alternatively, the Blaschke metric arises via a hyperbolic affine sphere over the universal cover of M, obtained from a Monge–Ampère equation solved by Cheng–Yau [CY77, CY86]; see [Lof10] for a survey. Our first main result complements these pictures by relating the Blaschke metric to a variational principle intrinsic to the projective structure:

Theorem A. *Let $(M, \mathfrak{p})$ be a closed oriented properly convex projective surface of negative Euler characteristic. Then the Blaschke metric of $\mathfrak{p}$ is a critical point for $\mathcal{E}$.*

We now turn to conformal variations of $\mathcal{E}$. In two dimensions, solving the variational equations for conformal variations is equivalent to solving a Poisson equation, and this allows us to prove:

Theorem B. *Let $(M, \mathfrak{p})$ be a closed oriented projective surface. Then every conformal structure on M contains a conformally critical metric, unique up to homothety.*

A surprising aspect of $\mathcal{E}$ is its close connection to the Yamabe equation [Yam60] and to the Einstein–Hilbert functional, which assigns to a metric g its volume-normalised total scalar curvature $\mathrm{Vol}_M(g)^{(2-n)/n} \int_M S_g d\mu_g$. To explain this connection, we use [Met20, Thm. 2.4], which shows that the choice of a conformal structure $[g]$ on a projective manifold $(M, \mathfrak{p})$ distinguishes a unique torsion-free connection ${}^{[g]}\nabla$ preserving $[g]$ – a so-called *Weyl connection* – and a 1-form $A_{[g]}$ on M with values in the endomorphisms of TM so that ${}^{[g]}\nabla + A_{[g]}$ lies in $\mathfrak{p}$.

For a metric g on $(M, \mathfrak{p})$, we define the *conformal defect* as

$$V_g = \frac{(n+1)(n-2)}{(n+2)}|A_{[g]}|_g^2 - \mathrm{tr}_g \operatorname{Ric}\left({}^{[g]}\nabla\right),$$

where $\operatorname{Ric}\left({}^{[g]}\nabla\right)$ denotes the Ricci curvature of ${}^{[g]}\nabla$. The motivation for introducing V_g is its relevance in finding conformally critical metrics for $\mathcal{E}$. Recall that, for $n \geqslant 3$ the conformal Laplacian of g is defined as

$$L_g := 4\left(\frac{n-1}{n-2}\right)\Delta_g + S_g.$$

Using V_g, we define the *projective-conformal Laplacian* $\mathcal{L}_g = L_g + V_g$ and show that finding a metric in $[g]$ that is conformally critical for $\mathcal{E}$ amounts to solving an equation of Yamabe type

$$\mathcal{L}_g u = C u^{(n+2)/(n-2)}$$

for some constant C and smooth positive function u on M.

Under a conformal change $g \mapsto \tilde{g} = \exp(2f)g$, the conformal defect changes as $V_{\tilde{g}} = \exp(-2f)V_g$. As a consequence of this, it follows that the *projective scalar curvature*

$$\mathcal{S}_g := S_g + V_g$$

transforms exactly as the scalar curvature under a conformal change $g \mapsto \tilde{g}$. Furthermore, we have the identity

$$\mathcal{E}(g) = \frac{n+2}{(n+1)(n-2)} \operatorname{Vol}_M(g)^{(2-n)/n} \int_M \mathcal{S}_g d\mu_g,$$

showing that our functional closely resembles the Einstein–Hilbert functional. We define the *conformal locus* of $[g]$ as $\Sigma_{[g]} := \left\{p \in M \mid V_g(p) = 0\right\}$. Note that $\Sigma_{[g]}$ is a conformal invariant of $[g]$. Building on the solution [Tru68, Aub76, Sch84] of the Yamabe problem [Yam60] and the theory of elliptic equations of Yamabe type (see in particular [DH05]), we obtain:

Theorem C. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geqslant 3$ and $[g]$ a conformal structure on M. Suppose that either $\Sigma_{[g]} = M$, or that $n \geqslant 4$ and there exists $p_0 \in M$ with $V_{\tilde{g}}(p_0) < 0$ for some (and hence any) $\tilde{g} \in [g]$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Drawing further on the analogy with the Yamabe equation, it is tempting to speculate that on a closed oriented projective manifold $(M, \mathfrak{p})$, every conformal equivalence class $[g]$ contains a conformally critical metric for $\mathcal{E}$. Some of the cases left open by Theorem C appear to be within reach of the localisation theory for elliptic equations of Yamabe type developed by Druet and Hebey [DH05]. Others – most notably the borderline regime in which $V_g \geqslant 0$ with $\Sigma_{[g]} \neq \emptyset$ – parallel the situation in which the classical Yamabe problem was settled by Schoen [Sch84] via the positive mass theorem of

Schoen and Yau [SY79, SY81]. We hope to return to these unresolved cases in future work.

Acknowledgements

The author is grateful to Rod Gover, Luca Giudici and Tobias Weth for helpful conversations.

2. Setup

2.1. Projective structures

Let M be a closed oriented smooth n-manifold with $n \geqslant 2$. Recall that the difference of two torsion-free connections ∇ and ∇' on TM is a section of the vector bundle $E = S^2(T^*M) \otimes TM$. For a section $\Phi = (\Phi^i_{jk})$ of E we define $\mathrm{Con}(\Phi) \in \Omega^1$ by the pointwise tensor contraction

$$[\mathrm{Con}(\Phi)]_i = \Phi^k_{ik}.$$

For $\alpha = (\alpha_i) \in \Omega^1$ we obtain a section $\mathrm{Sym}(\alpha)$ of E by the rule

$$[\mathrm{Sym}(\alpha)]^i_{jk} = \delta^i_j \alpha_k + \delta^i_k \alpha_j.$$

Observe that $\mathrm{Con}(\mathrm{Sym}(\alpha)) = (n+1)\alpha$.

For a section Φ of E we denote by Φ_0 its part in the kernel of the above contraction, that is,

$$\Phi_0 := \Phi - \tfrac{1}{n+1}\,\mathrm{Sym}\left(\mathrm{Con}(\Phi)\right). \tag{2.1}$$

In particular, we have

$$(\mathrm{Sym}(\alpha))_0 = \mathrm{Sym}(\alpha) - \tfrac{1}{(n+1)}\,\mathrm{Sym}(\mathrm{Con}(\mathrm{Sym}(\alpha))) = \mathrm{Sym}(\alpha) - \mathrm{Sym}(\alpha) = 0$$

for all $\alpha \in \Omega^1$, so that

$$E \simeq E_0 \oplus T^*M,$$

where E_0 denotes the contraction-free part of E.

Definition 2.1. Two torsion-free connections ∇ and ∇' on TM are called *projectively equivalent* if every geodesic of ∇ can be reparametrised to become a geodesic of ∇'. A *projective structure* on M is an equivalence class $\mathfrak{p}$ of projectively equivalent torsion-free connections on TM. A manifold M equipped with a projective structure $\mathfrak{p}$ will be called a *projective manifold*.

Remark 2.2. By a geodesic of ∇ we mean an immersed curve $\gamma : I \to M$ which satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$.

We have – see [Spi79] for a modern reference:

Lemma 2.3 (Cartan, Eisenhart, Weyl). *Two torsion-free connections ∇ and ∇' on TM are projectively equivalent if and only if $(\nabla - \nabla')_0 = 0$.*

2.2. A choice of metric

Besides a projective structure $\mathfrak{p}$, we now also fix a Riemannian metric g on M. The metric $g = (g_{ij})$ equips each real tensor bundle $E \to M$ with a bundle metric. Abusing notation, we denote this bundle metric on all the various tensor bundles by $\langle \cdot, \cdot \rangle_g$. For instance, if $\Phi = (\Phi^i_{jk})$ and $\Psi = (\Psi^i_{jk})$ are sections of $S^2(T^*M) \otimes TM$, then

$$\langle \Phi, \Psi \rangle_g = \Phi^i_{jk} g^{jb} g^{kc} g_{ia} \Psi^a_{bc},$$

where we write $g^\sharp = (g^{ij})$ for the dual metric. Correspondingly, we obtain an inner product on the set $\Gamma(E)$ of smooth sections of E by defining

$$\langle\langle s_1, s_2 \rangle\rangle_g = \int_M \langle s_1, s_2 \rangle_g d\mu_g,$$

where $d\mu_g$ denotes the volume form of g and $s_1, s_2 \in \Gamma(E)$. As usual, we write $|s|_g := \sqrt{\langle s, s \rangle_g}$ and likewise $\|s\|_g := \sqrt{\langle\langle s, s \rangle\rangle_g}$ for all $s \in \Gamma(E)$. For what follows, we record the identities

$$|\Phi|^2_{\exp(2f)g} = \mathrm{e}^{-2f} |\Phi|^2_g \tag{2.2}$$

and

$$d\mu_{\exp(2f)g} = \mathrm{e}^{nf} d\mu_g, \tag{2.3}$$

where $f \in C^\infty$ and $\Phi \in \Gamma(S^2(T^*M) \otimes TM)$.

The bundle E_0 with $E = S^2(T^*M) \otimes TM$ is irreducible as a $\mathrm{GL}^+(n, \mathbb{R})$-bundle, it is however not irreducible as a $\mathrm{SO}(n)$-bundle. Consider the metric trace

$$\mathrm{tr}_g : \Gamma(E_0) \to \Gamma(TM), \quad \Phi^i_{jk} \mapsto g^{jk} \Phi^i_{jk}$$

and the inclusion

$$\iota_g : \Gamma(TM) \to \Gamma(E_0), \quad X \mapsto (g \otimes X)_0$$

which satisfy

$$\mathrm{tr}_g \circ \iota_g = \tfrac{(n+2)(n-1)}{(n+1)} \mathrm{Id}_{\Gamma(TM)}.$$

Consequently, we can decompose $\Phi \in \Gamma(E_0)$ as

$$X = \frac{1}{m} \mathrm{tr}_g \Phi \quad \text{and} \quad A = \Phi - (g \otimes X)_0, \tag{2.4}$$

where $m = \frac{(n+2)(n-1)}{(n+1)}$. By construction, $\mathrm{tr}_g A$ vanishes identically. Furthermore, a simple computation shows that this decomposition is orthogonal. That is, if Z is a vector field on M and Φ a section of E_0 which satisfies $\mathrm{tr}_g \Phi \equiv 0$, then $\langle (g \otimes Z)_0, \Phi \rangle_g$ vanishes identically as well. Moreover, for the decomposition (2.4), we have

$$|\Phi|^2_g = |A|^2_g + m|X|^2_g, \tag{2.5}$$

since $|(g \otimes Y)_0|^2_g = m|Y|^2_g$ for every vector field Y on M.

Remark 2.4. Also observe that for $\Psi \in \Gamma(E_0)$ and a 1-form α on M we have

$$\langle \Psi, \mathrm{Sym}(\alpha) \rangle_g = 0, \tag{2.6}$$

as can be verified with a simple computation.

2.3. A projectively invariant functional

Let $\mathrm{Riem}(M)$ denote the set of Riemannian metrics on M. For a projective structure $\mathfrak{p}$ and orientation on M we consider the following projectively invariant non-negative functional

$$\mathcal{E} : \mathrm{Riem}(M) \to \mathbb{R}, \qquad g \mapsto \mathrm{Vol}_M(g)^{(2-n)/n} \int_M |(\nabla - {}^g\nabla)_0|^2_g d\mu_g,$$

where $\nabla \in \mathfrak{p}$ and ${}^g\nabla$ denotes the Levi-Civita connection of g. Note that the functional attains the value 0 if and only if there exists a Riemannian metric g whose Levi-Civita connection belongs to $\mathfrak{p}$. Also observe that if $\hat{\mathcal{F}} : \mathrm{Riem}(M) \to \mathbb{R}$ is a functional on the set of Riemannian metrics for which there exists a constant k such that $\hat{\mathcal{F}}(\mathrm{e}^{2c}g) = \mathrm{e}^{kc}\hat{\mathcal{F}}(g)$ for all constants c and metrics g, then the functional $\mathcal{F}$ defined by $\mathcal{F}(g) = \mathrm{Vol}_M(g)^{-k/n}\hat{\mathcal{F}}(g)$ is invariant under rescaling a metric by a positive constant. Since rescaling a metric by a positive constant does not change its Levi-Civita connection, (2.2) and (2.3) imply that

$$\mathcal{E}(cg) = \mathcal{E}(g)$$

for all positive constants c and all metrics $g \in \mathrm{Riem}(M)$.

Remark 2.5 (Notation)**.** The functional $\mathcal{E}$ depends on a choice of projective structure $\mathfrak{p}$ on M, as do a number of other objects we shall define. To keep the notation uncluttered, we leave this dependency implicit throughout.

3. Variational equations

Let V be a Fréchet space and $U \subset V$ an open subset. The partial derivative of a function $\mathcal{F} : U \to \mathbb{R}$ at $g \in U$ in the direction of $h \in V$ is defined by

$$\mathcal{F}'_g(h) := \lim_{t\to 0} \frac{1}{t}(\mathcal{F}(g + th) - \mathcal{F}(g)),$$

provided the limit exists. The set $\Gamma(S^2(T^*M))$ is a Fréchet space and the set of Riemannian metrics on M is an open subset thereof, hence $\mathrm{Riem}(M)$ is a Fréchet manifold whose tangent space at g is canonically isomorphic to $\Gamma(S^2(T^*M))$.

We say a Riemannian metric g on M is *critical* for $\mathcal{E}$ provided $\mathcal{E}'_g(h) = 0$ for all $h \in T_g\mathrm{Riem}(M) \simeq \Gamma(S^2(T^*M))$. Likewise, we say a metric g on M is *conformally critical* for $\mathcal{E}$, provided $\mathcal{E}'_g(h) = 0$ for all h of the form $h = fg$ with $f \in C^\infty$.

For a Riemannian metric g on M, $h \in \Gamma(S^2(T^*M))$ and $t \in \mathbb{R}$, we write $g_t = g + th$ as well as

$$\Phi_{g_t} := (\nabla - {}^{g_t}\nabla)_0 \qquad \text{and} \qquad \Phi_g := \Phi_{g_0}.$$

Let Γ denote the map which assigns to a Riemannian metric g its Levi-Civita connection and let $\Gamma'_g : \Gamma(S^2(T^*M)) \to \Gamma(E)$ denote its linearisation at $g \in \mathrm{Riem}(M)$

$$\Gamma'_g(h) = \lim_{t\to 0} \frac{1}{t}\left({}^{g_t}\nabla - {}^g\nabla\right).$$

Using Γ'_g we obtain a linear first order differential operator

$$\ell_g : \Gamma(S^2(T^*M)) \longrightarrow \Gamma(E_0), \quad h \mapsto \Big(\Gamma'_g(h)\Big)_0 .$$

We let

$$\ell_g^* : \Gamma(E_0) \longrightarrow \Gamma(S^2(T^*M))$$

denote the formal adjoint satisfying

$$\int_M \langle \Psi, \ell_g(h)\rangle_g d\mu_g = \int_M \langle \ell_g^*(\Psi), h\rangle_g d\mu_g \tag{3.1}$$

for all $h \in \Gamma(S^2(T^*M))$ and all $\Psi \in \Gamma(E_0)$.

Furthermore, for a section Ψ of E_0 we let $T_g(\Psi)$ denote the unique symmetric covariant 2-tensor field so that

$$\tfrac{\partial}{\partial t}\Big|_{t=0} |\Psi|^2_{g_t} d\mu_{g_t} = \langle T_g(\Psi), h\rangle_g d\mu_g$$

for all $h \in \Gamma(S^2(T^*M))$. In analogy with the physics literature, we call $T_g(\Psi)$ the *stress-energy tensor* of Ψ. The integrand $|\Phi_{g_t}|^2_{g_t} d\mu_{g_t}$ depends on t through two distinct quantities: the section Φ_{g_t} and the metric pairing together with the volume form $|\cdot|^2_{g_t} d\mu_{g_t}$. The chain rule therefore gives

$$\tfrac{\partial}{\partial t}\Big|_{t=0} |\Phi_{g_t}|^2_{g_t} d\mu_{g_t} = \tfrac{\partial}{\partial t}\Big|_{t=0} |\Phi_g|^2_{g_t} d\mu_{g_t} + \tfrac{\partial}{\partial t}\Big|_{t=0} |\Phi_{g_t}|^2_g d\mu_g, \tag{3.2}$$

where in the first term on the right the section is frozen at Φ_g, and in the second the pairing and volume are frozen at g. By the definition of the stress-energy tensor, the first term equals $\langle T_g(\Phi_g), h\rangle_g d\mu_g$. By the Leibniz rule applied to the inner product, the second term equals $2\langle \Phi_g, \tfrac{\partial}{\partial t}\big|_{t=0} \Phi_{g_t}\rangle_g d\mu_g$. We thus obtain

$$\tfrac{\partial}{\partial t}\Big|_{t=0} |\Phi_{g_t}|^2_{g_t} d\mu_{g_t} = \langle T_g(\Phi_g), h\rangle_g d\mu_g + 2\langle \Phi_g, \tfrac{\partial}{\partial t}\Big|_{t=0} \Phi_{g_t}\rangle_g d\mu_g. \tag{3.3}$$

Writing

$$\Phi_{g_t} = (\nabla - {}^g\nabla - ({}^{g_t}\nabla - {}^g\nabla))_0,$$

we observe that the difference $\nabla - {}^g\nabla$ is independent of t. Setting $H := \Gamma'_g(h)$, the definition of Γ'_g gives

$$\tfrac{\partial}{\partial t}\Big|_{t=0} ({}^{g_t}\nabla - {}^g\nabla) = H,$$

and since $\ell_g(h) = H_0$ by definition of ℓ_g, we conclude

$$\tfrac{\partial}{\partial t}\Big|_{t=0} \Phi_{g_t} = -H_0 = -\ell_g(h).$$

Substituting into (3.3) thus yields

$$\tfrac{\partial}{\partial t}\Big|_{t=0} |\Phi_{g_t}|^2_{g_t} d\mu_{g_t} = \langle T_g(\Phi_g), h\rangle_g d\mu_g - 2\langle \Phi_g, \ell_g(h)\rangle_g d\mu_g. \tag{3.4}$$

Now consider

$$\hat{\mathcal{E}}(g) := \int_M |(\nabla - {}^g\nabla)_0|^2_g d\mu_g = \int_M |\Phi_g|^2_g d\mu_g = \|\Phi_g\|^2_g$$

so that, interchanging $\frac{\partial}{\partial t}\big|_{t=0}$ with $\int_M$ and integrating (3.4) over M, the first two equalities below follow; the third uses the defining property (3.1) of ℓ_g^* to push ℓ_g across the inner product:

$$\hat{\mathcal{E}}_g'(h) = \frac{\partial}{\partial t}\Big|_{t=0} \int_M |\Phi_{g_t}|^2_{g_t} d\mu_{g_t} = \int_M \langle T_g(\Phi_g), h\rangle_g - 2\langle \Phi_g, \ell_g(h)\rangle_g d\mu_g$$
$$= \int_M \langle T_g(\Phi_g) - 2\ell_g^*(\Phi_g), h\rangle_g d\mu_g.$$

Thinking of the volume of M with respect to a metric as a functional $\mathrm{Vol}_M : \mathrm{Riem}(M) \to \mathbb{R}$, we have the standard identity

$$(\mathrm{Vol}_M)_g'(h) = \frac{1}{2}\int_M (\mathrm{tr}_g h) d\mu_g = \frac{1}{2}\int_M \langle g, h\rangle_g d\mu_g.$$

Since $\mathcal{E}(g) = \mathrm{Vol}_M(g)^{(2-n)/n}\hat{\mathcal{E}}(g)$, the product rule gives

$$\mathcal{E}_g'(h) = \mathrm{Vol}_M(g)^{(2-n)/n}\hat{\mathcal{E}}_g'(h) + \tfrac{2-n}{n}\mathrm{Vol}_M(g)^{(2-n)/n-1}\hat{\mathcal{E}}(g)\,(\mathrm{Vol}_M)_g'(h).$$

Substituting the formula for $(\mathrm{Vol}_M)_g'(h)$ above and using $\hat{\mathcal{E}}(g) = \|\Phi_g\|_g^2$, we obtain

$$\mathcal{E}_g'(h) = \mathrm{Vol}_M(g)^{(2-n)/n}\left[\hat{\mathcal{E}}_g'(h) - \left(\frac{n-2}{2n}\right)\frac{\|\Phi_g\|_g^2}{\mathrm{Vol}_M(g)}\int_M \langle g, h\rangle_g d\mu_g\right].$$

Combining this with the expression for $\hat{\mathcal{E}}_g'(h)$ derived above, and setting $k := \left(\frac{n-2}{2n}\right)\frac{\|\Phi_g\|_g^2}{\mathrm{Vol}_M(g)}$, we conclude

$$\mathcal{E}_g'(h) = \mathrm{Vol}_M(g)^{(2-n)/n}\int_M \langle T_g(\Phi_g) - 2\ell_g^*(\Phi_g) - kg,\, h\rangle_g d\mu_g. \tag{3.5}$$

By the fundamental lemma of the calculus of variations, $\mathcal{E}_g'(h) = 0$ for all $h \in \Gamma(S^2(T^*M))$ if and only if the symmetric tensor $T_g(\Phi_g) - 2\ell_g^*(\Phi_g) - kg$ vanishes identically. We thus have:

Proposition 3.1. *A Riemannian metric g is critical for $\mathcal{E}$ if and only if*

$$T_g(\Phi_g) - 2\ell_g^*(\Phi_g) = kg,$$

where the constant k is given by $k = \left(\frac{n-2}{2n}\right)\frac{\|\Phi_g\|_g^2}{\mathrm{Vol}_M(g)}$.

We also need:

Lemma 3.2. *For $\Psi \in \Gamma(E_0)$, the stress-energy tensor $T_g(\Psi) \in \Gamma(S^2(T^*M))$ is given by*

$$T_g(\Psi) = \frac{1}{2}|\Psi|_g^2 g + \Psi \circledast_g \Psi,$$

where

$$(\Psi \circledast_g \Psi)_{ij} = \Psi^k_{uv}\Psi^a_{bc}g_{ia}g_{jk}g^{ub}g^{vc} - 2\Psi^k_{jc}\Psi^a_{ib}g_{ak}g^{bc}.$$

In particular

$$\mathrm{tr}_g\left(T_g(\Psi)\right) = \left(\tfrac{n}{2} - 1\right)|\Psi|_g^2. \tag{3.6}$$

Proof. Recall the standard identities

$$\tfrac{\partial}{\partial t}\Big|_{t=0} d\mu_{g_t} = \frac{1}{2}(\mathrm{tr}_g h) d\mu_g \qquad \text{and} \qquad \tfrac{\partial}{\partial t}\Big|_{t=0} \left(g_t^\sharp\right)^{ij} = -h^{ij}$$

where we raise indices of h with the metric g. Then

$$|\Psi|^2_{g_t} d\mu_{g_t} = \Psi^k_{ij}\Psi^a_{bc}\,(g_{ak} + t h_{ak})\left(g^{ib} - t h^{ib} + \mathcal{O}(t^2)\right) \left(g^{jc} - t h^{jc} + \mathcal{O}(t^2)\right)\left(1 + \frac{1}{2} t g_{pq} h^{pq} + \mathcal{O}(t^2)\right) d\mu_g.$$

From this we calculate

$$\tfrac{\partial}{\partial t}\Big|_{t=0} |\Psi|^2_{g_t} d\mu_{g_t} = \Big[\Psi^k_{uv}\Psi^a_{bc} g_{ia} g_{jk} g^{ub} g^{vc} - 2\Psi^k_{jc}\Psi^a_{ib} g_{ak} g^{bc} + \tfrac{1}{2}|\Psi|^2_g g_{ij}\Big] h^{ij} d\mu_g,$$

so that

$$\tfrac{\partial}{\partial t}\Big|_{t=0} |\Psi|^2_{g_t} d\mu_{g_t} = \langle T_g(\Psi), h\rangle_g d\mu_g,$$

as claimed. The identity (3.6) follows from an elementary computation which we omit. □

Lemma 3.3. *For* $g \in \mathrm{Riem}(M)$ *the operator*

$$\ell_g^* : \Gamma(E_0) \longrightarrow \Gamma(S^2(T^*M))$$

is given by

$$\Psi^i_{jk} \mapsto \tfrac{1}{2}\left({}^g\nabla_k \Psi^k_{ij} - g^{uv} g_{ik}\, {}^g\nabla_u \Psi^k_{jv} - g^{uv} g_{jk}\, {}^g\nabla_u \Psi^k_{iv}\right).$$

Proof. We need to show that for $g \in \mathrm{Riem}(M)$, $h \in \Gamma(S^2(T^*M))$ and $\Psi \in \Gamma(E_0)$, we have

$$\int_M \langle \Psi, \ell_g(h)\rangle_g d\mu_g = \int_M \langle \ell_g^*(\Psi), h\rangle_g d\mu_g.$$

With our notation $H = \Gamma_g'(h)$, an elementary computation gives

$$H^k_{ij} = \frac{1}{2} g^{kl}\left({}^g\nabla_i h_{jl} + {}^g\nabla_j h_{il} - {}^g\nabla_l h_{ij}\right).$$

Recall from (2.1) that

$$H = H_0 + \tfrac{1}{(n+1)}\,\mathrm{Sym}(\mathrm{Con}(H)).$$

Since $\Psi \in \Gamma(E_0)$ and $\ell_g(h) = H_0$, we have

$$\langle \Psi, H\rangle_g = \langle \Psi, H_0\rangle_g + \tfrac{1}{(n+1)}\langle \Psi, \mathrm{Sym}(\mathrm{Con}(H))\rangle_g = \langle \Psi, H_0\rangle_g = \langle \Psi, \ell_g(h)\rangle_g,$$

where we use (2.6). Contracting $g_{ka} g^{al} = \delta^l_k$ gives

$$\langle \Psi, H\rangle_g = \frac{1}{2}\Psi^k_{ij} g^{ib} g^{jc}\left({}^g\nabla_b h_{ck} + {}^g\nabla_c h_{bk} - {}^g\nabla_k h_{bc}\right).$$

The first two terms contribute equally (swap $i \leftrightarrow j$, $b \leftrightarrow c$ and use $\Psi^k_{ij} = \Psi^k_{ji}$), so that

$$\langle \Psi, H\rangle_g = \Psi^k_{ij} g^{ib} g^{jcg} \nabla_b h_{ck} - \frac{1}{2}\Psi^k_{ij} g^{ib} g^{jcg} \nabla_k h_{bc}.$$

Consider the vector field $Y = (Y^l)$ on M with

$$Y^l = g^{il} g^{jc} \Psi^k_{ij} h_{ck}.$$

Since $\int_M \mathrm{div}_g Y \, d\mu_g = 0$ by Stokes' theorem, we obtain

$$\int_M g^{ib} g^{jc} \nabla_b (\Psi^k_{ij} h_{ck}) d\mu_g = 0$$

and hence

$$\int_M g^{ib} g^{jc} \Psi^{kg}_{ij} \nabla_b h_{ck} \, d\mu_g = - \int_M g^{ib} g^{jc} ({}^g\nabla_b \Psi^k_{ij}) \, h_{ck} \, d\mu_g.$$

Like-wise we obtain

$$-\frac{1}{2} \int_M \Psi^k_{ij} g^{ib} g^{jcg} \nabla_k h_{bc} \, d\mu_g = \frac{1}{2} \int_M g^{ib} g^{jc} ({}^g\nabla_k \Psi^k_{ij}) \, h_{bc} \, d\mu_g.$$

In the second integral, substituting $h_{bc} = g_{bp} g_{cq} h^{pq}$ gives $\frac{1}{2}({}^g\nabla_k \Psi^k_{pq}) \, h^{pq}$. In the first, substituting $h_{ck} = g_{cp} g_{kq} h^{pq}$ gives $-g^{ub} g_{kq} ({}^g\nabla_b \Psi^k_{up}) \, h^{pq}$. Symmetrising in p, q and relabelling summation indices (using $\Psi^k_{up} = \Psi^k_{pu}$) yields

$$[\ell^*_g(\Psi)]_{pq} = \frac{1}{2}\left({}^g\nabla_k \Psi^k_{pq} - g^{uv} g_{pk} {}^g\nabla_u \Psi^k_{qv} - g^{uv} g_{qk} {}^g\nabla_u \Psi^k_{pv}\right). \qquad \square$$

3.1. Decomposing Φ_g

Decomposing Φ_g as in (2.4)

$$X_g := \frac{1}{m} \operatorname{tr}_g \Phi_g \qquad \text{and} \qquad A_g := \Phi_g - (g \otimes X_g)_0$$

it turns out that A_g depends only on the conformal equivalence class $[g]$ of g. To this end let $f \in C^\infty$ and consider $\tilde{g} = \exp(2f) g$. Using the identity (see [Bes87, Thm. 1.159])

$$^{\tilde{g}}\nabla = {}^g\nabla - g \otimes \operatorname{grad}_g f + \operatorname{Sym}(\mathrm{d}f), \tag{3.7}$$

we obtain

$$\Phi_{\tilde{g}} = (\nabla - {}^{\tilde{g}}\nabla)_0 = \Phi_g + (g \otimes \operatorname{grad}_g f)_0 = \Phi_g + g \otimes \operatorname{grad}_g f - \frac{1}{(n+1)} \operatorname{Sym}(\mathrm{d}f).$$

where we use that $\operatorname{Con}(g \otimes \operatorname{grad}_g f) = \mathrm{d}f$. Hence we compute

$$\begin{aligned} X_{\tilde{g}} &= \frac{1}{m} \operatorname{tr}_{\tilde{g}} \Phi_{\tilde{g}} = \mathrm{e}^{-2f}\left(X_g + \frac{n}{m} \operatorname{grad}_g f - \frac{2}{(n+1)m} \operatorname{grad}_g f\right) \\ &= \mathrm{e}^{-2f}\left(X_g + \operatorname{grad}_g f\right). \end{aligned} \tag{3.8}$$

This also gives

$$\begin{aligned} A_{\tilde g} &= \Phi_{\tilde g} - (\tilde g \otimes X_{\tilde g})_0 \\ &= \Phi_g + g \otimes \operatorname{grad}_g f - \tfrac{1}{(n+1)} \operatorname{Sym}(df) - \left(g \otimes (X_g + \operatorname{grad}_g f)\right)_0 \\ &= A_g + g \otimes \operatorname{grad}_g f - \tfrac{1}{(n+1)} \operatorname{Sym}(df) - (g \otimes \operatorname{grad}_g f)_0 \\ &= A_g - \tfrac{1}{(n+1)} \operatorname{Sym}(df) + \tfrac{1}{(n+1)} \operatorname{Sym}(\operatorname{Con}(g \otimes \operatorname{grad}_g f)) = A_g. \end{aligned}$$

We will henceforth write $A_{[g]}$ instead of A_g. The conformal invariance of $A_{[g]}$ was previously observed in [**Met20**, Thm. 2.4]. We included the argument for the convenience of the reader.

For what follows we recall the standard identity for a vector field Y on M

$$\operatorname{div}_{\tilde g} Y = \operatorname{div}_g Y + nY(f),$$

which, together with (3.8), gives

$$\operatorname{div}_{\tilde g} X_{\tilde g} = e^{-2f} \left(\operatorname{div}_g X_g + (n-2) X_g(f) - \Delta_g f + (n-2)|df|_g^2\right), \tag{3.9}$$

where $\Delta_g = -\operatorname{div}_g \operatorname{grad}_g$.

3.2. Conformal variations

As an immediate corollary of Proposition 3.1 we have:

Corollary 3.4. *A metric g is conformally critical for $\mathcal{E}$ if and only if*

$$\operatorname{div}_g(\operatorname{tr}_g \Phi_g) = \left(\frac{n}{2} - 1\right)\left(|\Phi_g|_g^2 - \frac{\|\Phi_g\|_g^2}{\operatorname{Vol}_M(g)}\right). \tag{3.10}$$

Proof. Observe that we have

$$\operatorname{tr}_g\left(\ell_g^*(\Phi_g)\right) = \tfrac{1}{2} \operatorname{div}_g(\operatorname{tr}_g \Phi_g),$$

where

$$\operatorname{div}_g(\operatorname{tr}_g \Phi_g) = {}^g\nabla_i g^{jk} (\Phi_g)^i_{jk}.$$

Furthermore, recall (3.6)

$$\operatorname{tr}_g(T_g(\Phi_g)) = \left(\tfrac{n}{2} - 1\right) |\Phi_g|_g^2.$$

For $h = fg$ with $f \in C^\infty$, we obtain

$$\begin{aligned} \mathcal{E}_g'(fg) &= \operatorname{Vol}_M(g)^{(2-n)/n} \int_M f \operatorname{tr}_g \left(T_g(\Phi_g) - 2\ell_g^*(\Phi_g)\right) - knf d\mu_g \\ &= \operatorname{Vol}_M(g)^{(2-n)/n} \int_M f \left[\left(\frac{n}{2} - 1\right) |\Phi_g|_g^2 - \operatorname{div}_g(\operatorname{tr}_g \Phi_g) - kn\right] d\mu_g, \end{aligned}$$

and the claim follows. □

4. The surface case

4.1. Example: Projective structures from holomorphic cubic differentials

Let (M,g) be an oriented Riemannian 2-manifold and let $J : TM \to TM$ denote counter clockwise rotation by $\pi/2$ with respect to the orientation and metric g. We consider a cubic differential C on the Riemann surface (M,J). Its real part is a totally symmetric $(0,3)$ tensor field $A = (A_{ijk})$ on M that is totally trace-free with respect to g, that is, $g^{ij}A_{ijk}$ vanishes identically. Define $\alpha = (\alpha^i_{jk}) \in \Gamma(E)$ by $\alpha^i_{jk} = g^{il}A_{ljk}$ and consider the projective structure $\mathfrak{p}$ arising from ${}^g\nabla + \alpha$. By construction, $\alpha \in \Gamma(E_0)$ and $\Phi_g = \alpha$. Applying Proposition 3.1, the variational equations become $T_g(\alpha) = 2\ell^*_g(\alpha)$. By (3.6), the tensor field $T_g(\alpha)$ is totally trace-free with respect to g and hence encodes a quadratic differential on (M,J). The map T_g thus gives rise to a quadratic form on $\Gamma(K^3_{M,J})$ with values in $\Gamma(K^2_{M,J})$, where $K_{M,J}$ denotes the canonical bundle of (M,J). Unsurprisingly, this quadratic form is trivial (as can be checked with a direct computation), hence $T_g(\alpha)$ vanishes identically and the variational equations simplify to $\ell^*_g(\alpha) = 0$. A computation gives $2\ell^*_g(\alpha) = -\operatorname{div}_g \alpha$ with $(\operatorname{div}_g \alpha)_{ij} = {}^g\nabla_k\alpha^k_{ij}$. This last condition is well known to be equivalent to the holomorphicity of C. From this we obtain:

Theorem A. *Let $(M,\mathfrak{p})$ be a closed oriented properly convex projective surface of negative Euler characteristic. Then the Blaschke metric of $\mathfrak{p}$ is a critical point for $\mathcal{E}$.*

Proof. By [**Lof01, Lab07**], on a closed oriented surface M of negative Euler characteristic every *properly convex projective structure* $\mathfrak{p}$ arises from a unique pair (g,C) with C holomorphic via the above construction. That is, the torsion-free connection ${}^g\nabla+\alpha$ on TM is a representative connection of $\mathfrak{p}$. Additionally, the pair satisfies $K_g = -1+2|C|^2_g$, where K_g denotes the Gauss curvature of g. The metric g is known as the *Blaschke metric* of $\mathfrak{p}$. The above computations then show that $T_g(\alpha) = 0$ and $\ell^*_g(\alpha) = 0$ by the holomorphicity of C, so the Blaschke metric g is critical for $\mathcal{E}$. □

4.2. Existence and uniqueness of conformally critical metrics

Corollary 3.4 implies that for $n = 2$ a Riemannian metric g is conformally critical if and only if $\operatorname{div}_g X_g = 0$. Using (3.9), looking for a conformally critical metric $\tilde{g} = \mathrm{e}^{2f}g$ in the conformal equivalence class of g thus gives the PDE

$$\operatorname{div}_{\tilde{g}} X_{\tilde{g}} = \mathrm{e}^{-2f}\left(\operatorname{div}_g X_g - \Delta_g f\right) = 0.$$

Hence we obtain:

Theorem B. *Let $(M,\mathfrak{p})$ be a closed oriented projective surface. Then every conformal structure on M contains a conformally critical metric, unique up to homothety.*

Proof. Recall that for $u \in C^\infty$ the Poisson equation $\Delta_g f = u$ on a closed oriented Riemannian manifold (M, g) has a smooth solution, unique up to adding a constant, provided $\int_M u d\mu_g = 0$. In our case $u = \mathrm{div}_g X_g$, hence the claim is a consequence of Stokes' theorem. □

5. The projective Yamabe equation

5.1. The projective-conformal Laplacian

From now on we assume $n \geqslant 3$. Consider a closed oriented conformal n-manifold $(M, [g])$. Recall that the Yamabe problem asks whether there exists a metric $\tilde{g} \in [g]$ whose scalar curvature $S_{\tilde{g}}$ is constant. For $f \in C^\infty$ the scalar curvature of the metric $\tilde{g} = \exp(2f)g$ is

$$S_{\tilde{g}} = \mathrm{e}^{-2f}\left(S_g + 2(n-1)\Delta_g f - (n-1)(n-2)|\mathrm{d}f|_g^2\right),$$

where $\Delta_g f = -\,\mathrm{div}_g\,\mathrm{grad}_g\, f$. Writing $\tilde{g} = u^{4/(n-2)}g$ for a positive function u, the previous equation becomes

$$4\left(\frac{n-1}{n-2}\right)\Delta_g u + S_g u = u^{(n+2)/(n-2)}S_{\tilde{g}}.$$

Finding a metric $\tilde{g} \in [g]$ with constant scalar curvature C thus amounts to solving $L_g u = Cu^{(n+2)/(n-2)}$, where $L_g = 4\left(\frac{n-1}{n-2}\right)\Delta_g + S_g$ denotes the so-called *conformal Laplacian*.

We now return to the problem of finding a conformally critical metric on a closed oriented projective manifold of dimension $n \geqslant 3$. Writing

$$\mathcal{S}_g := \frac{2(n+1)}{(n+2)}\left[\left(\frac{n}{2}-1\right)|\Phi_g|_g^2 - \mathrm{div}_g(\mathrm{tr}_g\,\Phi_g)\right] \tag{5.1}$$

the equation (3.10) is equivalent to the statement that a metric g is conformally critical for $\mathcal{E}$ if and only if $\mathcal{S}_g$ is constant.

Lemma 5.1. *Let $f \in C^\infty$. Under a conformal change $g \mapsto \tilde{g} = \mathrm{e}^{2f}g$, the function $\mathcal{S}_g$ transforms as the scalar curvature, that is,*

$$\mathcal{S}_{\tilde{g}} = \mathrm{e}^{-2f}(\mathcal{S}_g + 2(n-1)\Delta_g f - (n-1)(n-2)|\mathrm{d}f|_g^2),$$

where $\Delta_g f = -\,\mathrm{div}_g \mathrm{grad}_g f$.

Proof. From the formula

$$\Phi_{\tilde{g}} = \Phi_g + g \otimes \mathrm{grad}_g\, f - \frac{1}{(n+1)}\mathrm{Sym}(\mathrm{d}f)$$

we compute

$$\begin{aligned}|\Phi_{\tilde{g}}|_{\tilde{g}}^2 &= \mathrm{e}^{-2f}|\Phi_g + (g \otimes \mathrm{grad}_g f)_0|_g^2 \\ &= \mathrm{e}^{-2f}\left(|\Phi_g|_g^2 + 2mX_g(f) + |(g \otimes \mathrm{grad}_g f)_0|_g^2\right) \\ &= \mathrm{e}^{-2f}\left(|\Phi_g|_g^2 + 2mX_g(f) + m|\mathrm{d}f|_g^2\right),\end{aligned} \tag{5.2}$$

where we have used the identities

$$|\operatorname{Sym}(\mathrm{d}f)|^2_g = 2(n+1)|\mathrm{d}f|^2_g \quad \text{and} \quad |g \otimes \operatorname{grad}_g f|^2_g = n|\mathrm{d}f|^2_g$$

as well as

$$\langle g \otimes \operatorname{grad}_g f, \operatorname{Sym}(\mathrm{d}f)\rangle_g = 2|\mathrm{d}f|^2_g.$$

Now using (3.9) and (5.2) we obtain

$$\begin{aligned}
\mathcal{S}_{\tilde{g}} &= \frac{(n-1)(n-2)}{m}|\Phi_{\tilde{g}}|^2_{\tilde{g}} - 2(n-1)\operatorname{div}_{\tilde{g}} X_{\tilde{g}}\\
&= \mathrm{e}^{-2f}\left[\mathcal{S}_g + 2(n-1)(n-2)X_g(f) + (n-1)(n-2)|\mathrm{d}f|^2_g\right.\\
&\quad \left.-2(n-1)(n-2)X_g(f) + 2(n-1)\Delta_g f - 2(n-1)(n-2)|\mathrm{d}f|^2_g\right]\\
&= \mathrm{e}^{-2f}\left(\mathcal{S}_g + 2(n-1)\Delta_g f - (n-1)(n-2)|\mathrm{d}f|^2_g\right),
\end{aligned}$$

as claimed. □

Finding a metric $\tilde{g}$ in a conformal equivalence class $[g]$ which is conformally critical for $\mathcal{E}$ is thus equivalent to solving an equation of Yamabe type $\mathcal{L}_g u = Cu^{(n+2)/(n-2)}$, where

$$\mathcal{L}_g := 4\left(\frac{n-1}{n-2}\right)\Delta_g + \mathcal{S}_g.$$

Remark 5.2. We will refer to $\mathcal{L}_g$ as the *projective-conformal Laplacian*.

5.2. The projective scalar curvature

As an immediate consequence of Stokes' theorem and definition (5.1) we have

Proposition 5.3. *Let g be a Riemannian metric on the closed oriented projective manifold $(M, \mathfrak{p})$ of dimension $n \geqslant 3$. Then*

$$\int_M \mathcal{S}_g \, d\mu_g = \hat{m}\, \|\Phi_g\|^2_g \geqslant 0, \tag{5.3}$$

where $\hat{m} = \frac{(n+1)(n-2)}{(n+2)}$, with equality if and only if $\mathfrak{p}$ contains the Levi-Civita connection of g.

Denoting by $\sigma(\mathcal{L}_g)$ the spectrum of $\mathcal{L}_g$, we also conclude:

Proposition 5.4. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geqslant 3$ and g a Riemannian metric on M. Then $\sigma(\mathcal{L}_g)$ is non-negative and moreover, contains 0 if and only if $\mathfrak{p}$ is metrisable by a metric $\tilde{g}$ in the conformal equivalence class of g.*

Proof. For every positive function u on M, writing $\tilde{g} = u^{4/(n-2)}g$ and using $\mathcal{S}_{\tilde{g}} = u^{-(n+2)/(n-2)}\mathcal{L}_g u$ as well as $d\mu_{\tilde{g}} = u^{2n/(n-2)}d\mu_g$, we obtain

$$\int_M u\,\mathcal{L}_g u\, d\mu_g = \int_M u^{2n/(n-2)}\mathcal{S}_{\tilde{g}}\, d\mu_g = \int_M \mathcal{S}_{\tilde{g}}\, d\mu_{\tilde{g}} = \hat{m}\,\|\Phi_{\tilde{g}}\|^2_{\tilde{g}} \geqslant 0.$$

Since, by standard elliptic PDE theory, the first eigenfunction of the self-adjoint elliptic operator $\mathcal{L}_g$ can be chosen positive, it follows that the first eigenvalue λ_1 of $\mathcal{L}_g$ satisfies $\lambda_1 \geqslant 0$ and hence the full spectrum is non-negative. If $\lambda_1 = 0$, the corresponding positive eigenfunction u satisfies $\mathcal{L}_g u = 0$, so that $\|\Phi_{\tilde{g}}\|^2_{\tilde{g}} = 0$ for $\tilde{g} = u^{4/(n-2)}g$, meaning $\mathfrak{p}$ contains the Levi-Civita connection of $\tilde{g} \in [g]$. Conversely, if $\tilde{g} = u^{4/(n-2)}g$ metrises $\mathfrak{p}$, then $\Phi_{\tilde{g}} = 0$ and hence $\mathcal{L}_g u = u^{(n+2)/(n-2)}\mathcal{S}_{\tilde{g}} = 0$, so that 0 is in the spectrum of $\mathcal{L}_g$. □

Because of the analogy to the scalar curvature, we will refer to $\mathcal{S}_g$ as the *projective scalar curvature* of g. For what follows it is advantageous to express $\mathcal{S}_g$ in terms of the usual scalar curvature of g, the "scalar curvature" of a certain Weyl connection and $A_{[g]}$. It follows from (3.7) and (3.8) that the torsion-free connection

$$ {}^{[g]}\nabla := {}^{g}\nabla + g \otimes X_g - \mathrm{Sym}(X_g^\flat) \tag{5.4}$$

depends only on the conformal equivalence class of g. Here $\flat$ is taken with respect to g. The connection ${}^{[g]}\nabla$ on TM is a *Weyl connection* for $[g]$, that is, it is torsion-free and its parallel transport maps are angle preserving with respect to $[g]$. In fact, all Weyl connections for $[g]$ are of the form (5.4) for some metric $g \in [g]$ and some vector field X. For a pair (g, X) we write

$$S_{g,X} := \mathrm{tr}_g \,\mathrm{Ric}\left({}^{[g]}\nabla\right),$$

where $\mathrm{Ric}\left({}^{[g]}\nabla\right)$ denotes the Ricci curvature of the Weyl connection ${}^{[g]}\nabla$ determined by (g, X). A standard calculation – see Section A – gives:

$$S_{g,X} = S_g + 2(n-1)\,\mathrm{div}_g X - (n-2)(n-1)|X|^2_g, \tag{5.5}$$

where S_g denotes the scalar curvature of g.

Proposition 5.5. *For the projective scalar curvature we have*

$$\mathcal{S}_g = S_g - S_{g,X_g} + \hat{m}|A_{[g]}|^2_g.$$

Proof. Using the definition (5.1), (5.5) and (2.5), we compute

$$\begin{aligned}
\mathcal{S}_g &= \frac{2(n+1)}{(n+2)}\left[\left(\frac{n}{2}-1\right)|\Phi_g|^2_g - \mathrm{div}_g(\mathrm{tr}_g\,\Phi_g)\right] \\
&= \hat{m}|\Phi_g|^2_g - 2(n-1)\,\mathrm{div}_g(X_g) \\
&= \hat{m}|A_{[g]}|^2_g + \hat{m}m|X_g|^2_g - 2(n-1)\,\mathrm{div}_g(X_g) \\
&= S_g - \left(S_g + 2(n-1)\,\mathrm{div}_g(X_g) - (n-2)(n-1)|X_g|^2_g\right) + \hat{m}|A_{[g]}|^2_g \\
&= S_g - S_{g,X_g} + \hat{m}|A_{[g]}|^2_g,
\end{aligned}$$

where we also used that $\hat{m}m = (n-2)(n-1)$. □

Remark 5.6. In terms of X_g and $A_{[g]}$ we thus have

$$\mathcal{S}_g = \hat{m}|A_{[g]}|^2_g + (n-2)(n-1)|X_g|^2_g - 2(n-1)\,\mathrm{div}_g(X_g).$$

6. Existence of conformally critical metrics

For a Riemannian metric g on the projective manifold $(M, \mathfrak{p})$ we define:

Definition 6.1. The *conformal defect* of g on $(M, \mathfrak{p})$ is the difference of the projective – and standard scalar curvature:

$$V_g := \mathcal{S}_g - S_g = \hat{m}|A_{[g]}|_g^2 - S_{g,X_g}.$$

Remark 6.2.

(i) Notice that $\mathcal{L}_g = L_g + V_g$, so that V_g measures the defect of the projective-conformal Laplacian to agree with the conformal Laplacian.

(ii) Recall that we define the conformal locus of g as

$$\Sigma_{[g]} = \left\{p \in M \,|\, V_g(p) = 0\right\}.$$

Under a conformal change $\tilde{g} = e^{2f}g$, the conformal defect satisfies $V_{\tilde{g}} = e^{-2f}V_g$. In particular, the conformal locus does only depend on the conformal equivalence class of g.

6.1. Reduction to the Yamabe problem

In the special case where the conformal locus of $[g]$ is all of M, the projective-conformal Laplacian and the conformal Laplacian agree, hence finding a conformally critical metric for $\mathcal{E}$ amounts to solving the Yamabe problem. The affirmative solution to the Yamabe problem [Yam60] by Trudinger [Tru68], Aubin [Aub76], and Schoen [Sch84] thus implies (we also refer to [LP87] and [DH05] for surveys):

Theorem 6.3. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geqslant 3$. Suppose the conformal structure $[g]$ on M satisfies $\Sigma_{[g]} = M$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Proof. By assumption $V_g \equiv 0$, so $\mathcal{L}_g = L_g$ and the criticality equation reduces to the Yamabe equation, which has a positive smooth solution by the work of Yamabe, Trudinger, Aubin and Schoen. □

6.2. Negative conformal defect

We treat in this subsection the case where the conformal defect V_g takes a negative value at some point of M. Since $V_{\tilde{g}} = e^{-2f}V_g$ for $\tilde{g} = e^{2f}g$, the hypothesis that V_g is negative at a given point is independent of the choice of representative metric in $[g]$.

Theorem 6.4. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geqslant 4$ and $[g]$ a conformal structure on M. Suppose there exists $p_0 \in M$ such that $V_{\tilde{g}}(p_0) < 0$ for some (and hence any) $\tilde{g} \in [g]$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Proof. By Section 5.1, finding a metric $\tilde{g} \in [g]$ which is conformally critical for $\mathcal{E}$ is equivalent to finding a smooth positive function u on M that solves $\mathcal{L}_g u = C u^{(n+2)/(n-2)}$ for some constant $C \geqslant 0$. Setting $h := \frac{n-2}{4(n-1)} \mathcal{S}_g$ and $\lambda := \frac{n-2}{4(n-1)} C$, this rewrites as the elliptic equation of Yamabe type

$$\Delta_g u + h\, u = \lambda\, u^{(n+2)/(n-2)}$$

in the sense of [DH05]. The hypothesis $V_g(p_0) < 0$ becomes

$$h(p_0) < \tfrac{n-2}{4(n-1)} S_g(p_0),$$

that is, the linear term of the equation drops below the linear term of the standard Yamabe equation at the point p_0. Existence of a smooth positive solution then follows from [DH05, Thm. 4.1] and the discussion thereafter. □

Combining Theorem 6.3 and Theorem 6.4 we have:

Theorem C. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geqslant 3$ and $[g]$ a conformal structure on M. Suppose that either $\Sigma_{[g]} = M$, or that $n \geqslant 4$ and there exists $p_0 \in M$ with $V_{\tilde{g}}(p_0) < 0$ for some (and hence any) $\tilde{g} \in [g]$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Remark 6.5. Theorem 6.4 covers, in particular, the empty-locus subcase $V_g < 0$ on M. The complementary empty-locus subcase $V_g > 0$ on M remains open.

Appendix A. Ricci curvature of a Weyl connection

Since the formula for the Ricci curvature of a Weyl connection does not seem to be easy to locate in the literature, we include the calculation.

Let ∇ be a torsion-free connection on TM with curvature tensor $R = (R^i_{jkl})$ and let $\Phi = (\Phi^i_{jk})$ be a 1-form on M with values in the endomorphisms of TM satisfying $\Phi(v)(w) = \Phi(w)(v)$ for all $v, w \in T_pM$ and all $p \in M$. Then the torsion-free connection $\hat{\nabla} = \nabla + \Phi$ has curvature

$$\hat{R}^i_{jkl} = R^i_{jkl} + \nabla_k \Phi^i_{lj} - \nabla_l \Phi^i_{kj} + \Phi^u_{lj}\Phi^i_{ku} - \Phi^u_{kj}\Phi^i_{lu}$$

as follows from an elementary computation. In particular, for the Ricci curvature $\hat{R}_{jk} = \hat{R}^l_{jlk}$ of $\hat{\nabla}$ we obtain

$$\hat{R}_{jk} = R_{jk} + \nabla_l \Phi^l_{jk} - \nabla_k \Phi^l_{jl} + \Phi^u_{jk}\Phi^l_{ul} - \Phi^u_{jl}\Phi^l_{ku}.$$

For (g, X) consider the Weyl connection

$$\hat{\nabla} := {}^g\nabla + g \otimes X - \mathrm{Sym}(X^\flat).$$

Take $\Phi = g \otimes X - \mathrm{Sym}(X^\flat)$ and $\nabla = {}^g\nabla$ so that

$$\Phi^i_{jk} = X^i g_{jk} - \delta^i_j \xi_k - \delta^i_k \xi_j,$$

where $\xi_i = g_{ij} X^j$. Then we have

$$\nabla_l \Phi^l_{jk} = \nabla_l \left(X^l g_{jk} - \delta^l_j \xi_k - \delta^l_k \xi_j \right) = \mathrm{div}_g(X) g_{jk} - \nabla_j \xi_k - \nabla_k \xi_j$$

Likewise, we obtain

$$\nabla_k \Phi^l_{jl} = -n\nabla_k \xi_j$$

as well as

$$\Phi^u_{jk}\Phi^l_{ul} = 2n\xi_j\xi_k - n|X|^2_g g_{jk}$$

and

$$\Phi^u_{jl}\Phi^l_{ku} = (n+2)\xi_j\xi_k - 2|X|^2_g g_{jk}.$$

From this we compute for the Ricci curvature $\hat{R}_{jk}$ of the Weyl connection $\tilde{\nabla}$

$$\begin{aligned}\hat{R}_{jk} &= R_{jk} + \mathrm{div}_g(X)g_{jk} - \nabla_j\xi_k - \nabla_k\xi_j + n\nabla_k\xi_j \\ &\quad + 2n\xi_j\xi_k - n|X|^2_g g_{jk} - (n+2)\xi_j\xi_k + 2|X|^2_g g_{jk} \\ &= R_{jk} + (n-2)(\nabla_{(j}\xi_{k)} + \xi_j\xi_k - |X|^2_g g_{jk}) + \mathrm{div}_g(X)g_{jk} - \tfrac{n}{2}(\mathrm{d}\xi)_{jk},\end{aligned}$$

where R_{jk} denotes the Ricci curvature of ∇, round brackets on indices denote symmetrisation and square brackets denote anti-symmetrisation. In particular, the symmetric and antisymmetric parts of the Ricci curvature are

$$\hat{R}_{(jk)} = R_{jk} + (n-2)\left(\nabla_{(j}\xi_{k)} + \xi_j\xi_k - |X|^2_g g_{jk}\right) + \mathrm{div}_g(X)g_{jk}$$

and

$$\hat{R}_{[jk]} = -\tfrac{n}{2}(\mathrm{d}\xi)_{jk}.$$

From this we compute

$$S_{g,X} = S_g + 2(n-1)\,\mathrm{div}_g(X) - (n-2)(n-1)|X|^2_g,$$

where $S_{g,X} := \mathrm{tr}_g\,\mathrm{Ric}(\hat{\nabla})$ and S_g denotes the scalar curvature of g.

References

[Aub76] T. Aubin, The scalar curvature, in *Differential geometry and relativity*, *Mathematical Phys. and Appl. Math., Vol. 3*, Reidel, Dordrecht, 1976, pp. 5–18. 3, 16

[Bes87] A. L. Besse, *Einstein manifolds*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* **10**, Springer-Verlag, Berlin, 1987. 10

[BDE09] R. Bryant, M. Dunajski, M. Eastwood, Metrisability of two-dimensional projective structures, *J. Differential Geom.* **83** no. 3 (2009), 465–499. arXiv:0801.0300 1

[CY77] S. Y. Cheng, S. T. Yau, On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x,u)$, *Comm. Pure Appl. Math.* **30** no. 1 (1977), 41–68. 2

[CY86] S. Y. Cheng, S.-T. Yau, Complete affine hypersurfaces. I. The completeness of affine metrics, *Comm. Pure Appl. Math.* **39** no. 6 (1986), 839–866. 2

[DH05] O. Druet, E. Hebey, Elliptic equations of Yamabe type, *IMRS Int. Math. Res. Surv.* no. 1 (2005), 1–113. 3, 16, 17

[Hit87] N. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc. (3)* **55** no. 1 (1987), 59–126. 2

[Hit92] N. Hitchin, Lie groups and Teichmüller space, *Topology* **31** no. 3 (1992), 449–473. 2

[Lab07] F. Labourie, Flat projective structures on surfaces and cubic holomorphic differentials, *Pure Appl. Math. Q.* **3** no. 4, Special Issue: In honor of Grigory Margulis. Part 1 (2007), 1057–1099. arXiv:math/0611250 2, 12

[LP87] J. M. Lee, T. H. Parker, The Yamabe problem, *Bull. Amer. Math. Soc. (N.S.)* **17** no. 1 (1987), 37–91. 16

[Lof10] J. Loftin, Survey on affine spheres, in *Handbook of geometric analysis, No. 2, Adv. Lect. Math. (ALM)* **13**, Int. Press, Somerville, MA, 2010, pp. 161–191. arXiv:0809.1186 2

[Lof01] J. C. Loftin, Affine spheres and convex $\mathbb{RP}^n$-manifolds, *Amer. J. Math.* **123** no. 2 (2001), 255–274. 2, 12

[Met20] T. Mettler, Extremal conformal structures on projective surfaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **20** no. 4 (2020), 1621–1663. arXiv:1510.01043 2, 11

[Sch84] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Differential Geom.* **20** no. 2 (1984), 479–495. 3, 16

[SY79] R. Schoen, S. T. Yau, On the proof of the positive mass conjecture in general relativity, *Comm. Math. Phys.* **65** no. 1 (1979), 45–76. 3

[SY81] R. Schoen, S. T. Yau, Proof of the positive mass theorem. II, *Comm. Math. Phys.* **79** no. 2 (1981), 231–260. 3

[Spi79] M. Spivak, *A comprehensive introduction to differential geometry. Vol. II*, second ed., Publish or Perish Inc., Wilmington, Del., 1979. 4

[Tru68] N. S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **22** (1968), 265–274. 3, 16

[Wie18] A. Wienhard, An invitation to higher Teichmüller theory, in *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, World Sci. Publ., Hackensack, NJ, 2018, pp. 1013–1039. arXiv:1803.06870 2

[Yam60] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* **12** (1960), 21–37. 2, 3, 16

Faculty of Mathematics and Computer Science, UniDistance Suisse, Brig, Switzerland

Email address: `thomas.mettler@fernuni.ch; mettler@math.ch`