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Twistor geometric aspects of projective surfaces

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ABSTRACT. This note summarizes results obtained while the author was a member of the SPP 2026 "Geometry at Infinity" by DFG, focusing on the interplay between projective differential geometry and twistor theory on surfaces. Specifically, we provide a new twistorial perspective on the classical metrizability problem for projective structures, showing that a projective surface admits a compatible Riemannian metric if and only if the associated conformal structure embeds as both a holomorphic curve into its twistor space as well as a pseudo-holomorphic curve into the twistor space with a suitably modified almost complex structure. We also establish that while every projective surface locally admits a compatible Weyl connection, no such connection exists globally on non-hyperbolic properly convex projective surfaces — in particular, such surfaces are not metrizable. Additionally, we explore the duality between Finsler 2-spheres of constant curvature with all geodesics closed and positive Besse–Weyl structures on weighted projective lines. This correspondence, realized through the notion of the dual of a path geometry, provides new examples of such Finsler metrics via deformations of the Veronese embedding.

1. Introduction

Projective differential geometry studies the notion of a *projective structure* on a smooth manifold. A projective structure is a differential geometric structure with finite dimensional automorphism group, and fits into the general framework of so-called *parabolic geometries*. See [**4, 13, 14**] for additional background. Concretely, a projective structure on a smooth manifold N is an equivalence class $\mathfrak{p}$ of torsion-free connections on its tangent bundle TN, where two such connections are projectively equivalent if they have the same geodesics up to parametrization. A manifold equipped with a projective structure is called a *projective manifold*. A homogeneous model for a projective manifold is given by projective space, where the projective structure arises from the geodesics of its standard metric. Similar to a Riemannian manifold being a curved generalisation of Euclidean space, a projective manifold may be thought of as being a curved generalisation of projective space.

A question that has stimulated much activity in projective differential geometry in recent years is that of *metrisability*. Given a projective structure $\mathfrak{p}$, is there a Riemannian metric g whose Levi-Civita connection ${}^g\mathrm{D}$ is an

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element of $\mathfrak{p}$? The existence problem for g can be turned into an overdetermined linear partial differential equation of finite type which in general does not admit a solution, see [**10**] and [**12, 15, 19, 20, 21**] for related recent work.

In recent years there have been significant advances in the area of higher Teichmüller theory, see [**46**] for a survey and [**24**] for Hitchin's landmark contribution. This has generated substantial interplay between Riemann surfaces equipped with holomorphic data and geometric structures on surfaces, or geometric structures on suitable bundles over surfaces. A common feature of these geometric structures is that they give rise to a flat connection on a bundle over the surface and correspondingly to a representation of its fundamental group. A prototypical example of a geometric structure on a smooth surface M arising in higher Teichmüller theory is that of a *flat properly convex* projective structure. Here a projective manifold $(N, \mathfrak{p})$ is called flat if it is locally diffeomorphic to an open subset of $\mathbb{R}^n$ in such a way that the geodesics of $\mathfrak{p}$ are mapped onto segments of straight lines. Furthermore, $(N, \mathfrak{p})$ is called *properly convex* if it arises as a quotient of a properly convex open set $\tilde{N} \subset \mathbb{RP}^n$ by a group $\Gamma \subset \mathrm{PSL}(n+1, \mathbb{R})$ of projective transformations which acts discretely and properly discontinuously. Properly convex projective surfaces include hyperbolic Riemann surfaces as a special case, but, as we will see in Section 2.1, non-hyperbolic examples are never metrisable. For background about properly convex projective manifolds we refer the reader to [**17, 26, 27, 34, 35**].

A particularly rich connection exists between two-dimensional projective differential geometry and twistor theory. The theory of twistors was invented by Nobel laureate Roger Penrose [**42, 43**] in an attempt to unify gravity and quantum mechanics. While unsuccessful, the theory has since evolved into a very active branch of pure mathematics [**25, 30, 44**] and theoretical physics [**1, 2, 18**], just to name a few recent contributions. As an influential paper on the mathematical side, let us mention [**3**] in which the authors – among other things – show how to construct the twistor space of an oriented Riemannian 4-manifold. Inspired by Hitchin's [**23**] and LeBrun's [**33**] construction of holomorphic projective structures by twistorial techniques, [**16**] and [**41**] adapt techniques from [**3**] to construct a twistor space for smooth projective structures. We will sketch its construction below.

In [**31**] (see also [**32**]) LeBrun–Mason introduce the notion of a Zoll projective structure. A projective structure $\mathfrak{p}$ is called *Zoll* if the image of every maximal geodesic of $\mathfrak{p}$ is an embedded circle. They show that the only closed surfaces admitting Zoll projective structures are S^2 and $\mathbb{RP}^2$. Generalizing the rigidity result for Zoll metrics on $\mathbb{RP}^2$, they also show that the standard projective structure is the only example of a Zoll projective structure on $\mathbb{RP}^2$, up to diffeomorphism. On the other hand, LeBrun–Mason use twistorial techniques to show that the standard projective structure on the 2-sphere admits deformations within the class of Zoll projective structures. This is achieved by compactifying the twistor space in the case of a Zoll projective 2-sphere. The resulting compact surface is biholomorphic to $\mathbb{CP}^2$ and totally real embeddings of $\mathbb{RP}^2$ into the compactified twistor

space $\mathbb{CP}^2$ that are sufficiently close to the standard real linear embedding are shown to correspond to Zoll projective structures on S^2. This allowed LeBrun–Mason to prove that there is a 5-dimensional deformation space of the standard Zoll projective structure on S^2. Similarly, twistorial methods were vital in Bryant's classification of projectively flat Finsler 2-spheres of constant curvature [**11**], further demonstrating the effectiveness of the twistor approach in surface geometry.

These applications of twistorial techniques by LeBrun–Mason and Bryant illustrate how the complex geometry of twistor spaces can reveal profound insights about the underlying geometric structures. The power of this approach has continued to inspire new developments in the field. Recently, Paternain et al. introduced and investigated the so-called *transport twistor space* of an oriented Riemannian 2-manifold (M, g). The transport twistor space of (M, g) is a certain branched cover of the twistor space of the oriented projective surface $(M, [^g\mathrm{D}])$. Interest in the transport twistor space stems from the fact that its complex geometry neatly encodes the geodesic flow of the Riemannian 2-manifold (M, g). This allows one to investigate various questions in Riemannian dynamics (and geometric inverse problems) in the context of twistor geometry. We refer the reader to [**5, 6, 7, 8**] for details.

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2. Overview of results

In this section, we present two main results concerning geometric structures on surfaces. First, we revisit the classical problem of when a projective structure admits a compatible Riemannian metric, providing a new perspective through twistor theory. Second, we explore the correspondence between Finsler 2-spheres of constant curvature and Besse–Weyl structures on weighted projective lines.

2.1. Metrisability of projective surfaces revisited

We start by providing some additional background about projective structures. We consider the case of projective structures on a smooth surface M. Recall that the difference of two torsion-free connections ∇ and ∇' on TM is a section of the vector bundle $E = S^2(T^*M) \otimes TM$. For a section $\Phi = (\Phi^i_{jk})$ of E we define $\operatorname{Con}\Phi \in \Omega^1_M$ by the pointwise tensor contraction

$$(\operatorname{Con}\Phi)_i = \Phi^k_{ik},$$

where we employ the summation convention.

For $\alpha = (\alpha_i) \in \Omega^1_M$ we obtain a section $\operatorname{Sym}\alpha$ of E by the rule

$$(\operatorname{Sym}\alpha)^i_{jk} = \delta^i_j\alpha_k + \delta^i_k\alpha_j.$$

Observe that $\mathrm{Con}(\mathrm{Sym}\,\alpha) = 3\alpha$. For a section Φ of E we denote by Φ_0 its part in the kernel of the above contraction, that is,

$$\Phi_0 = \Phi - \frac{1}{3}\,\mathrm{Sym}(\mathrm{Con}\,\Phi).$$

In particular, we obtain

$$(\mathrm{Sym}\,\alpha)_0 = \mathrm{Sym}\,\alpha - \frac{1}{3}\,\mathrm{Sym}(\mathrm{Con}(\mathrm{Sym}\,\alpha)) = \mathrm{Sym}\,\alpha - \mathrm{Sym}\,\alpha = 0 \tag{2.1}$$

for all $\alpha \in \Omega^1_M$ so that

$$E \simeq E_0 \oplus T^*M,$$

where E_0 denotes the contraction-free part of E.

Recall that two torsion-free connections ∇ and ∇' on TM are called *projectively equivalent* if ∇ and ∇' have the same geodesics up to parametrization. By a geodesic of ∇ we mean an immersed curve $\gamma \subset M$ which satisfies $\nabla_{\dot\gamma}\dot\gamma \equiv 0$. We have (see [**45**] for a modern reference):

Lemma 2.1 (Cartan, Eisenhart, Weyl)**.** *Two torsion-free connections ∇ and ∇' on TM are projectively equivalent if and only if $(\nabla - \nabla')_0 = 0$.*

Consequently, the set of smooth projective structures on M is an affine space whose associated vector space is given by the space of smooth sections of the rank 4 vector bundle E_0. It follows from the representation theory of $\mathrm{GL}(2,\mathbb{R})$ that as a $\mathrm{GL}(2,\mathbb{R})$-bundle, E_0 is canonically isomorphic to $S^3(T^*M) \otimes \Lambda^2(TM)$. Given a choice of orientation and Riemannian metric g on M, the bundle E_0 decomposes into irreducible $\mathrm{SO}(2)$-bundles $E_0 \simeq T^*M \oplus S^3_0(T^*M)$, where the latter summand denotes the totally symmetric (0,3) tensors on M that are trace-free with respect to g. The choice of a Riemannian metric g on M thus allows to encode a projective structure $\mathfrak{p}$ in terms of a triple (g, υ_g, A_g), where υ_g is a 1-form on M and A_g a totally symmetric (0,3) tensor on M that is trace-free with respect to g.

To make things more explicit, for a *projective surface* $(M, \mathfrak{p})$ and a choice of Riemannian metric g on M, consider the vector field

$$Y_g = \tfrac{3}{4}\,\mathrm{tr}_g\,(\nabla - {}^g\mathrm{D})_0\,,$$

and the endomorphism-valued 1-form

$$\varphi_{[g]} = (\nabla - {}^g\mathrm{D} - g \otimes Y_g)_0\,,$$

where $\nabla \in \mathfrak{p}$ is any representative connection. These two tensor fields are of interest because the torsion-free connection

$$\nabla = {}^g\mathrm{D} + g \otimes Y_g - \mathrm{Sym}(\upsilon_g) + \varphi_{[g]}$$

is a representative connection of $\mathfrak{p}$, where $\upsilon_g \in \Omega^1_M$ denotes the g-dual 1-form to Y_g. In fact, both ∇ and $\varphi_{[g]}$ only depend on the conformal equivalence class $[g]$ of g. The totally symmetric and trace-free $(0,3)$ tensor A_g is obtained by lowering the upper index of $\varphi_{[g]}$ with g. Notice that the projective structure $\mathfrak{p}$ is metrisable by the metric g if and only if both $\varphi_{[g]}$ and υ_g vanish identically.

The choice of metric g on M gives rise to two closely related connections, which – in general – are not elements of $\mathfrak{p}$, namely

$$ {}^{[g]}\mathrm{D} = {}^{g}\mathrm{D} + g \otimes Y_g - \mathrm{Sym}(\upsilon_g) \qquad \text{and} \qquad {}^{dA_g}\mathrm{D} = {}^{g}\mathrm{D} + \varphi_{[g]}. $$

The first connection ${}^{[g]}\mathrm{D}$ is a *Weyl connection* for $[g]$, that is, a torsion-free connection on TM whose parallel transport maps are angle preserving with respect to $[g]$. We will refer to ${}^{[g]}\mathrm{D}$ as the Weyl connection determined by $[g]$ and to the second connection ${}^{dA_g}\mathrm{D}$ as the *special connection* determined by g. Notice that ${}^{dA_g}\mathrm{D}$ preserves the area form dA_g of g and hence has holonomy in the special linear group $\mathrm{SL}(2,\mathbb{R})$.

In what follows, we will give a different perspective onto A_g and υ_g by interpreting a projective structure $\mathfrak{p}$ as a special case of a path geometry. A *path geometry* on an oriented surface M prescribes a path $\gamma \subset M$ for every oriented tangent direction. To make the notion concise, consider $\mathbb{S}M$, the circle bundle over M whose fibre $\mathbb{S}_pM$ at a point $p \in M$ consists of all oriented 1-dimensional linear subspaces of T_pM. The total space of the bundle $\nu : \mathbb{S}M \to M$ is equipped with a tautological co-orientable contact distribution C. For an oriented line $\ell \in \mathbb{S}M$ we let $C_\ell \subset T_\ell \mathbb{S}M$ consist of those tangent vectors which are mapped into ℓ under the differential ν' of the base point projection. A path geometry on M consists of a 1-dimensional distribution $P \to \mathbb{S}M$ which, together with the vertical distribution $L = \ker \nu'$, spans C. The paths of the path geometry P are its integral curves, projected down to M.

A convenient way to turn P into a more hands on object is to fix a Riemannian metric g on M. Doing so allows to identify $\mathbb{S}M$ with the principal right $\mathrm{SO}(2)$-bundle SM consisting of unit tangent vectors with respect to g. Explicitly, we map $\ell \in \mathbb{S}M$ to the unique unit length vector $u \in \ell$ which is positive with respect to the orientation of ℓ. Recall that SM is equipped with two canonical 1-forms $\omega_1, \omega_2 \in \Omega^1_{SM}$ defined by the rules $\omega_1(v) := g(u, \nu'(v))$ and $\omega_2(v) := g(Ju, \nu'(v))$, for all $u \in SM$ and all $v \in T_uSM$. Here J denotes counter-clockwise rotation by $\pi/2$ of a vector tangent to M with respect to the orientation of M and we also denote the basepoint projection of $SM \to M$ by ν. Notice that under the identification $\mathbb{S}M \simeq SM$ provided by the metric g, the contact distribution C is identified with the kernel of ω_2. The total space of the bundle SM is also equipped with the Levi-Civita connection form $\psi \in \Omega^1_{SM}$ of g and we have the structure equations

$$ \begin{aligned} \mathrm{d}\omega_1 &= -\omega_2 \wedge \psi, \\ \mathrm{d}\omega_2 &= -\psi \wedge \omega_1, \\ \mathrm{d}\psi &= -K_g \omega_1 \wedge \omega_2, \end{aligned} \tag{2.2} $$

where K_g denotes the (pullback to SM of the) *Gauss curvature* of g. The three 1-forms $(\omega_1, \omega_2, \psi)$ trivialise the cotangent bundle of SM and we denote by (X, H, V) the corresponding dual vector fields. The vector field X is the geodesic vector field of g, V is the infinitesimal generator of the $\mathrm{SO}(2)$-action and H is the horizontal vector field satisfying $H = [V, X]$.

Under the identification $\mathbb{S}M \to SM$ provided by a choice of Riemannian metric g on M, a path geometry P is identified with a line bundle on SM which is spanned by a vector field F on SM which can be written as $F = \kappa X + \tau H + \lambda V$ for smooth real-valued functions κ, τ, λ on SM. Since the contact distribution C containing P is identified with the kernel of ω_2, it follows that τ must vanish identically. Moreover, since P is transverse to the vertical direction of the basepoint projection, κ cannot vanish and hence we can assume – without losing generality – that the line bundle is spanned by a vector field of the form $F = X + \lambda V$. The flow ϕ of the vector field F is sometimes called a (generalized) *thermostat*. See for instance [**38**] for background.

Given a path geometry in terms of a vector field $F = X + \lambda V$ on SM, we may ask when there exists a projective structure $\mathfrak{p}$ on M so that the orbits of thermostat ϕ associated to F project to M to agree with the geodesics of $\mathfrak{p}$. In this case, we call the thermostat ϕ projective. In order to answer this question we introduce some further terminology.

Following [**22**], we use the volume form $\Theta = \omega_1 \wedge \omega_2 \wedge \psi$ on SM to define an inner product

$$\langle u, v \rangle = \int_{SM} u \bar{v} \, \Theta$$

for complex-valued functions u, v on SM and we denote by $L^2(SM)$ the corresponding space of square integrable complex-valued functions on SM. The structure equations (2.2) and Cartan's formula imply that all vector fields X, H, V preserve Θ. In particular, $-iV$ is densely defined and self-adjoint with respect to $\langle \cdot, \cdot \rangle$. Consequently, we have an orthogonal direct sum decomposition into the kernels $\mathcal{H}_m$ of the operators $m\mathrm{Id} + iV$

$$L^2(SM) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m. \tag{2.3}$$

The decomposition (2.3) may be thought of as a fibre-wise Fourier decomposition of a complex-valued function on SM and the fact that a choice of orientation and Riemannian metric g on M encodes a projective structure $\mathfrak{p}$ in terms of a triple (g, v_g, A_g) translates to the following statement:

Proposition 2.2 ([**39**])**.** *A thermostat ϕ associated to $X + \lambda V$ is projective if and only if $\lambda \in \mathcal{H}_{-1} \oplus \mathcal{H}_1 \oplus \mathcal{H}_{-3} \oplus \mathcal{H}_3$.*

Here the degree 3 part of λ encodes A_g and the degree 1 part of λ encodes v_g. We refer to [**39**] for further details.

Finding a Riemannian metric g on M so that $\mathfrak{p}$ is metrisable by the metric g thus amounts to finding g so that v_g vanishes identically and so that $\varphi_{[g]}$ vanishes identically. Perhaps surprisingly, the two partial differential equations $v_g \equiv 0$ and $\varphi_{[g]} \equiv 0$ admit a twistorial interpretation. To explain this we sketch the construction of the twistor space of an oriented projective surface and refer to [**16, 41**] for details: For an oriented smooth surface M, we consider the disk bundle $\pi : Z \to M$ whose fibre at a point $p \in M$ consists of the orientation compatible linear complex structure on T_pM. The choice of a torsion-free connection ∇ on TM equips Z with an almost complex

structure J. Namely, at $j \in Z$ we lift j horizontally and take a natural complex structure on each fibre vertically. It can be shown that J is always integrable and does only depend on the projective equivalence class $\mathfrak{p}$ of ∇, we thus denote it by $J_{\mathfrak{p}}$ and refer to the complex surface $(Z, J_{\mathfrak{p}})$ as the *twistor space* of the oriented projective surface $(M, \mathfrak{p})$. A conformal structure $[g]$ on the oriented surface gives rise to an orientation compatible linear complex structure on each tangent space. For $p \in M$ we define $j_p : T_pM \to T_pM$ by counter clockwise rotation by $\pi/2$ with respect to the orientation of T_pM. We can thus think of a conformal structure on M as a section $[g] : M \to Z$ and we have:

Theorem 2.3 ([**37**]). *Let $(M, \mathfrak{p})$ be an oriented projective surface and $[g]$ a conformal structure on M. Then the following statements are equivalent:*

(1) *$\varphi_{[g]}$ vanishes identically;*
(2) *$[g] : M \to (Z, J_{\mathfrak{p}})$ is a holomorphic curve;*
(3) *the Weyl connection ${}^{[g]}\mathrm{D}$ determined by $[g]$ is an element of $\mathfrak{p}$.*

Reversing the orientation on each fibre of $Z \to M$ yields another almost complex structure on Z which is however never integrable and not projectively invariant. A choice of volume form σ on $(M, \mathfrak{p})$ determines a unique representative connection ${}^{\sigma}\nabla \in \mathfrak{p}$ which preserves σ. We will denote the non-integrable almost complex structure on Z arising from ${}^{\sigma}\nabla$ by $\mathfrak{J}_{\mathfrak{p},\sigma}$. Denoting the area form of a metric g by dA_g, we also have:

Theorem 2.4 ([**37**]). *Let $(M, \mathfrak{p})$ be an oriented projective surface and g a Riemannian metric on M. Then the following statements are equivalent:*

(1) *υ_g vanishes identically;*
(2) *$[g] : M \to (Z, \mathfrak{J}_{\mathfrak{p},dA_g})$ is a pseudo-holomorphic curve;*
(3) *the special connection ${}^{dA_g}\mathrm{D}$ determined by g is an element of $\mathfrak{p}$.*

Combining the two previous propositions, we thus conclude that an oriented projective surface $(M, \mathfrak{p})$ is metrisable by the metric g on M if and only if $[g] : M \to (Z, J_{\mathfrak{p}})$ is a holomorphic curve and $[g] : M \to (Z, \mathfrak{J}_{\mathfrak{p},dA_g})$ is a pseudo-holomorphic curve. Here we say $[g] : M \to (Z, \mathfrak{J})$ is a (pseudo-)holomorphic curve if the image $\mathcal{M} = [g](M) \subset Z$ admits the structure of a (pseudo-)holomorphic curve. By admitting the structure of (pseudo-)holomorphic curve, we mean that $\mathcal{M}$ can be equipped with a complex structure J, so that the inclusion $\iota : \mathcal{M} \to Z$ is $(J, \mathfrak{J})$-linear, that is, satisfies $\mathfrak{J} \circ \iota' = \iota' \circ J$.

Local existence of (pseudo-)holomorphic curves into the twistor space follows from [**40**]. This allows to conclude that every oriented projective surface $(M, \mathfrak{p})$ locally admits a conformal structure $[g]$ so that the Weyl connection ${}^{[g]}\mathrm{D}$ determined by $[g]$ is an element of $\mathfrak{p}$. Likewise, locally it admits a Riemannian metric g so that the special connection ${}^{dA_g}\mathrm{D}$ determined by g is an element of $\mathfrak{p}$.

The aforementioned local existence results raise the question about global existence results for holomorphic curves into the twistor space. In general,

global existence of a holomorphic curve into the twistor space does not hold. Non-hyperbolic properly convex projective surfaces yield counterexamples:

Theorem 2.5 ([**39**]). *Let $(M, \mathfrak{p})$ be a closed oriented properly convex projective surface with $\chi(M) < 0$. Suppose there exists a conformal structure $[g]$ on M so that the Weyl connection ${}^{[g]}\mathrm{D}$ determined by $[g]$ is an element of $\mathfrak{p}$. Then $\mathfrak{p}$ is hyperbolic and ${}^{[g]}\mathrm{D}$ is the Levi-Civita connection of the hyperbolic metric.*

In particular, since metrisability of $\mathfrak{p}$ by a metric g implies $\varphi_{[g]} \equiv 0$, and hence that ${}^{[g]}\mathrm{D}$ is an element of $\mathfrak{p}$, it follows that non-hyperbolic properly convex projective surfaces are not globally metrisable.

2.2. Finsler 2-spheres of constant curvature

The twistor space of an oriented projective surface also plays a role in the context of Finsler metrics on the 2-sphere that have constant (positive) curvature. The reason for this is a notion of duality for path geometries. Following Bryant [**11**], a *generalized path geometry* consists of a 3-manifold N together with an ordered pair of one-dimensional distributions (P, L) which span a contact distribution. A generalized path geometry is sometimes also referred to as a *Legendrian contact structure*. In the generalized setting, the surface M is replaced with the leaf space of the foliation $\mathscr{L}$ determined by L and the leaf space of the foliation $\mathscr{P}$ determined by P can be thought of as the space of oriented paths of the generalized path geometry (N, P, L). We may reverse the order of (P, L) and thus consider the *dual* (L, P) of a generalized path geometry.

A Finsler metric on an oriented surface M naturally equips its unit tangent bundle $\Sigma \subset TM$ with a generalized path geometry. In the special case where all the geodesics of the Finsler metric are closed and the metric has constant positive curvature, the dual path geometry arises from a certain generalization of a Besse 2-orbifold [**28**] with positive curvature. Indeed, using [**9**], it is shown in [**29**] that the space of oriented geodesics of a Finsler 2-sphere of constant curvature and all geodesics closed is a *weighted projective line* $\mathbb{CP}(a_1, a_2)$ which comes equipped with a positive Besse–Weyl structure. By this we mean a Weyl connection whose symmetrized Ricci curvature is positive definite and with the property that the image of every maximal geodesic is an immersed circle. Conversely, having such a positive Besse-Weyl structure on a weighted projective line, its dual path geometry yields a Finsler metric on S^2 with constant positive curvature and all of whose geodesics are closed. More precisely:

Theorem 2.6 ([**29**]). *There is a one-to-one correspondence between Finsler structures on S^2 with constant curvature 1 and all geodesics closed on the one hand, and positive Besse–Weyl structures on the weighted projective line $\mathbb{CP}(a_1, a_2)$ with $c := \gcd(a_1, a_2) \in \{1, 2\}$, $a_1 \geqslant a_2$, $2|(a_1 + a_2)$ and $c^3|a_1 a_2$ on the other hand. More precisely,*

(1) *such a Finsler metric with shortest closed geodesic of length* $2\pi\ell \in (\pi, 2\pi]$, $\ell = p/q \in (\frac{1}{2}, 1]$, $\gcd(p,q) = 1$, *gives rise to a positive Besse–Weyl structure on* $\mathbb{CP}(a_1, a_2)$ *with* $a_1 = q$ *and* $a_2 = 2p - q$, *and*
(2) *a positive Besse–Weyl structure on such a* $\mathbb{CP}(a_1, a_2)$ *gives rise to such a Finsler metric on* S^2 *with shortest closed geodesic of length* $2\pi\left(\frac{a_1+a_2}{2a_1}\right) \in (\pi, 2\pi]$,

and these assignments are inverse to each other. Moreover, two such Finsler metrics are isometric if and only if the corresponding Besse–Weyl structures coincide up to a diffeomorphism.

The weighted projective space $\mathbb{CP}(a_1, \dots, a_n)$ is the quotient of $\mathbb{C}^n \setminus \{0\}$ by $\mathbb{C}^*$, where $\mathbb{C}^*$ acts with weights $(a_1, \dots, a_n) \in \mathbb{N}^n$, that is, by the rule

$$z \cdot (z_1, \dots, z_n) = (z^{a_1} z_1, \dots, z^{a_n} z_n)$$

for all $z \in \mathbb{C}^*$ and all $(z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{0\}$. The usual projective line $\mathbb{CP}^1$ corresponds to the case $a_1 = a_2 = 1$.

In the special case where we consider $\mathbb{CP}^1 = \mathbb{CP}(1,1)$ equipped with the projective structure arising from the Levi-Civita connection of the standard Fubini-Study metric, the resulting twistor space is $\mathbb{CP}^2 \setminus \mathbb{RP}^2$, see [**36**]. Here we think of $\mathbb{RP}^2$ as a subset of $\mathbb{CP}^2$ via its standard real linear embedding. Because of this, it is tempting to speculate that for a suitable generalization of the Fubini-Study metric on $\mathbb{CP}(a_1, a_2)$, the resulting twistor space can be holomorphically embedded into a weighted projective plane $\mathbb{CP}^2(b_1, b_2, b_3)$ for suitable weights $(b_1, b_2, b_3) \in \mathbb{N}^3$. This is indeed the case. To avoid case differentiations we will henceforth restrict to the case where the pair (a_1, a_2) is co-prime with $a_1 \geqslant a_2$ and both numbers odd. In this case it turns out that

$$(b_1, b_2, b_3) = (a_1, (a_1 + a_2)/2, a_2).$$

The holomorphic curve into the twistor space $Z \subset \mathbb{CP}^2$ arising from the Fubini-Study metric on $\mathbb{CP}(1,1)$ corresponds to the Veronese embedding. Using a suitable orbifold version of Theorem 2.3 allows to conclude that small enough deformations of the Veronese embedding of $\mathbb{CP}(a_1, a_2)$ into $\mathbb{CP}(a_1, (a_1 + a_2)/2, a_2)$ give examples of positive Besse-Weyl structures on $\mathbb{CP}(a_1, a_2)$. Via our duality result Theorem 2.6 these deformations also give new examples of Finsler 2-spheres of constant curvature and with all geodesics closed.

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