Induced para-Kähler Einstein metrics on cotangent bundles

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ABSTRACT. In earlier work we have shown that for certain geometric structures on a smooth manifold M of dimension n, one obtains a para-Kähler–Einstein metric on a manifold A of dimension $2n$ associated to the structure on M. The geometry also provides a family of diffeomorphisms between A and T^*M, so one can use this construction to obtain metrics on the cotangent bundle of M. In this short article, we discuss the relation of these metrics to Patterson–Walker metrics and derive explicit formulae for them in the cases of projective, conformal and Grassmannian structures.

1. Introduction

An *almost para-Kähler structure* on a $2n$-manifold N consists of a pseudo-Riemannian metric h of split-signature (n,n) and a symplectic form Ω such that the endomorphism $I : TN \to TN$ defined by $\Omega = h(I\cdot,\cdot)$ satisfies $I^2 = \mathrm{Id}_{TN}$. Almost para-Kähler structures are sometimes also called *almost bi-Lagrangian structures*, see for instance [3, 12, 13]. In [5], generalising the results from [11], we showed how to canonically construct an almost para-Kähler structure (h,Ω) on a manifold of dimension $2n$, which is naturally associated to a geometric structure (from a certain class) on an n-manifold M. The structures in question admit a description in terms of a so-called torsion-free $|1|$-graded parabolic geometry. This class of geometric structures includes in particular projective, conformal and Grassmannian (i.e. torsion-free almost Grassmannian) structures. What makes the construction remarkable is that h is always an *Einstein metric*.

The almost para-Kähler structure is defined on the total space of an affine bundle $\mu : A \to M$ whose sections can be interpreted as the *Weyl structures* of the parabolic geometry and whose associated vector bundle is the cotangent bundle $\nu : T^*M \to M$. In [5] it is also shown that the choice of a Weyl structure $s : M \to A$ induces a diffeomorphism $\varphi_s : T^*M \to A$ satisfying

$$(\varphi_s)^*\Omega = \mathrm{d}\tau - \nu^*(\mathrm{Alt}\,\mathbf{P}^s),$$

where τ denotes the tautological 1-form of T^*M and $\mathrm{Alt}\,\mathbf{P}^s$ the alternating part of the *Rho-tensor* $\mathbf{P}^s$ of s. This Rho-tensor is a curvature quantity associated to s which is an analog of the Ricci curvature.

The purpose of this short note is to relate the resulting metrics to classically known metrics on the cotangent bundle. In particular, we show – see Theorem 3.3

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below – that the pullback of the metric has a universal structure given by

$$(\varphi_s)^*h = h_{\vartheta_s} - \nu^*(\operatorname{Sym}\mathbf{P}^s) + \boldsymbol{q}, \tag{1.1}$$

where h_{ϑ_s} is the *Patterson–Walker metric* of the Weyl connection ϑ_s determined by s (see below for details). Further, $\operatorname{Sym}\mathbf{P}^s$ denotes the symmetric part of the Rho-tensor of s and $\boldsymbol{q}$ is a symmetric covariant 2-tensor field on T^*M which only depends on the underlying geometric structure and which is semibasic for the projection $\nu : T^*M \to M$. Moreover, $\boldsymbol{q}$ is homogeneous of degree 2 in the fibres of T^*M, that is, satisfies $(\mathcal{S}_t)^*\boldsymbol{q} = t^2\boldsymbol{q}$ where $\mathcal{S}_t : T^*M \to T^*M$ denotes scaling of a cotangent vector by the factor $t \in \mathbb{R}^*$.

We work out explicit expressions for (1.1) in the case of projective, conformal and Grassmannian structures. In the case of projective and conformal geometry, expressions for the Rho tensor are well-known, so this amounts to computing $\boldsymbol{q}$. For projective structures, we recover the result from [11], where it is shown that $\boldsymbol{q} = -\tau \otimes \tau$. Additionally, we show that in the case of a conformal structure $[g]$ on M, the tensor field $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = -\tau \otimes \tau + \frac{1}{2}|\cdot|^2_{g^\sharp}\nu^*g$$

where $g \in [g]$, $g^\sharp \in \Gamma(S^2(TM))$ denotes the dual metric of g and $|\cdot|^2_{g^\sharp} : T^*M \to \mathbb{R}$ is defined by $\xi \mapsto g^\sharp(\xi,\xi)$. Notice that $\boldsymbol{q}$ does only depend on $[g]$.

In the Grassmannian case, we derive a similar explicit description of $\boldsymbol{q}$ in terms of an "improved" version of the tautological one form obtained from the structure and we show how to compute the Rho tensor.

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2. Preliminaries

We start by briefly collecting some basic facts about soldering forms, Patterson–Walker metrics, $|1|$-graded parabolic geometries and Weyl structures. Throughout this article all manifolds and mappings are assumed to be smooth, that is, C^∞.

2.1. Soldering forms

Let M be an n-manifold and V a real n-dimensional vector space. We consider a right principal G_0-bundle $\pi_0 : \mathcal{G}_0 \to M$ for some Lie group G_0 and $\rho : G_0 \to \mathrm{GL}(V)$ a representation of G_0 on the vector space V. Suppose that we have a 1-form $\omega \in \Omega^1(\mathcal{G}_0, V)$ on $\mathcal{G}_0$ with values in V which is semibasic and ρ-equivariant. The former condition means that ω vanishes on all vectors of $T\mathcal{G}_0$ that are tangent to the fibres of π_0 and the latter condition means that ω satisfies $R_a^*\omega = \rho(a^{-1})(\omega)$ for all $a \in G_0$, where $R_a : \mathcal{G}_0 \to \mathcal{G}_0$ denotes the right action by a. Recall that such a form defines a 1-form $\boldsymbol{\omega} \in \Omega^1(M, E)$ on M with values in the vector bundle $\nu : \mathcal{G}_0 \times_\rho V =: E \to M$ associated to ρ. Indeed, for all $v \in TM$ the

element $\boldsymbol{\omega}(v) \in E$ is represented by $(u, \omega(\tilde{v})) \in \mathcal{G}_0 \times V$ for any choice of $u \in \mathcal{G}_0$ having the same basepoint as v and any $\tilde{v} \in T_u\mathcal{G}_0$ such that $\pi'_u(\tilde{v}) = v$, where $\pi'_u : T_u\mathcal{G}_0 \to T_{\pi(u)}M$ denotes the derivative of π at u. The 1-form $\omega \in \Omega^1(\mathcal{G}_0, V)$ is called a *soldering form* if $\boldsymbol{\omega}$ – thought of as a map $TM \to E$ – is a vector bundle isomorphism. Recall that this is equivalent to the fact that ω is *strictly horizontal* in the sense that its kernel in any point of $\mathcal{G}_0$ is the vertical subbundle.

2.2. The Patterson–Walker metric

We review the construction of the Patterson–Walker metric, adapted to our setting. To this end suppose, as above, that $\pi : \mathcal{G}_0 \to M$ is a right principal G_0-bundle equipped with a soldering form $\omega \in \Omega^1(\mathcal{G}_0, V)$ which is equivariant with respect to a representation $\rho : G_0 \to \mathrm{GL}(V)$. Assume in addition that ρ is infinitesimally effective, that is, the induced Lie algebra representation $\varrho : \mathfrak{g}_0 \to \mathfrak{gl}(V)$ is injective. Using ω, the tangent bundle of M can be identified with the bundle associated to ρ, that is, $TM \simeq \mathcal{G}_0 \times_\rho V$.

Now let $\vartheta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ be a principal G_0-connection on $\mathcal{G}_0$. Let us denote by V^* the dual space to V and by $\varrho^* : \mathfrak{g}_0 \to \mathfrak{gl}(V^*)$ the the dual representation to ϱ. Then on the product $\mathcal{G}_0 \times V^*$ we consider the V^*-valued 1-form

$$\zeta_\vartheta = \mathrm{d}\xi + (\varrho^* \circ \vartheta)(\xi),$$

where $\xi : \mathcal{G}_0 \times V^* \to V^*$ denotes the second projection. By construction, the V^*-valued 1-form is equivariant with respect to the dual representation $\rho^* : G_0 \to \mathrm{GL}(V^*)$ and it is easy to check that it is semibasic for the projection $\mathcal{G}_0 \times V^* \to T^*M \simeq \mathcal{G}_0 \times_{\rho^*} V^*$. Consequently, it represents a 1-form $\boldsymbol{\zeta}_\vartheta \in \Omega^1(T^*M, \nu^*T^*M)$ on T^*M with values in the pullback of the cotangent bundle $\nu : T^*M \to M$ of M. The pullback bundle ν^*T^*M is naturally a subbundle of $T^*(T^*M)$ and hence we may interpret $\boldsymbol{\zeta}_\vartheta$ as a 1-form on T^*M with values in $T^*(T^*M)$, or equivalently, as a section of $T^*(T^*M) \otimes T^*(T^*M)$. The symmetric part $h_\vartheta = \mathrm{Sym}\, \boldsymbol{\zeta}_\vartheta$ is a pseudo-Riemannian metric of split-signature (n, n) known as the *Patterson–Walker metric* associated to the connection ϑ, see [17] for the original construction and [14, 15] for recent applications.

2.3. Parabolic geometries

Recall that a Cartan geometry of type (G, P) for some Lie group G and closed subgroup $P \subset G$ is a pair $(\pi : \mathcal{G} \to M, \theta)$ consisting of a right principal P-bundle $\pi : \mathcal{G} \to M$ together with a *Cartan connection* θ taking values in the Lie algebra $\mathfrak{g}$ of G. We let $R : \mathcal{G} \times P \to \mathcal{G}$ denote the right action of P and we write $R_g = R(\cdot, g)$ for all $g \in P$ and $\iota_u = R(u, \cdot)$ for all $u \in \mathcal{G}$. The 1-form $\theta \in \Omega^1(\mathcal{G}, \mathfrak{g})$ being a *Cartan connection* means that for all $u \in \mathcal{G}$ the linear map $\theta_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism and moreover

$$(R^*\theta)_{(u,g)} = (\iota_u^*\theta)_g + (R_g^*\theta)_u = (\Upsilon_P)_g + \mathrm{Ad}(g^{-1}) \circ \theta_u, \tag{2.1}$$

for all $(u, g) \in \mathcal{G} \times P$, where Υ_P denotes the Maurer–Cartan form of P and $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ the adjoint action of G. In addition, the curvature 2-form $\Theta = \mathrm{d}\theta + \frac{1}{2}[\theta, \theta]$ of θ satisfies a further condition called *normality*. The details of this conditions are not important for our purposes, some consequences will be

discussed below. We will always assume that the Cartan geometry is *torsion-free*, that is, Θ has values in $\mathfrak{p} \subset \mathfrak{g}$, the Lie algebra of P.

Here we consider the special case of $|1|$-graded parabolic geometries. This means that G is assumed to be semisimple and that its Lie algebra $\mathfrak{g}$ is endowed with a so-called $|1|$-grading. This is a decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

into a direct sum of linear subspaces $\mathfrak{g}_i$ for $i = -1, 0, 1$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ with the convention that $\mathfrak{g}_\ell = \{0\}$ for $\ell = \pm 2$. Furthermore, no simple ideal of $\mathfrak{g}$ is allowed to be contained in $\mathfrak{g}_0$ and the Lie algebra $\mathfrak{p}$ of P satisfies $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In particular, this implies that $P \subset G$ is a parabolic subgroup in the sense of representation theory.

These properties have some nice consequences. First, the exponential map of $\mathfrak{g}$ – restricted to $\mathfrak{g}_1$ – is a diffeomorphism from $\mathfrak{g}_1$ onto a closed normal subgroup $P_+ \subset P$. Second, defining $G_0 \subset P$ to consist of those elements g so that the adjoint action $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$ preserves the grading of $\mathfrak{g}$, one can show that the Lie algebra of G_0 is $\mathfrak{g}_0$ and that G_0 is isomorphic to the quotient P/P_+. Third, every element g of P can be written as $g = g_0 \exp(Z)$ for unique elements $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$.

Since $P_+ \subset P$ is a normal subgroup, we obtain a principal $P/P_+ \simeq G_0$-bundle $\mathcal{G}/P_+ \to M$ whose total space we denote by $\mathcal{G}_0$ and whose basepoint projection we denote by π_0. We can project the values of the Cartan connection θ to $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$. Equivariancy of θ then easily implies that the result descends to a 1-form $\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1})$. Equivariancy of θ also implies that ω is equivariant with respect to the G_0-representation $\rho : G_0 \to \mathrm{GL}(\mathfrak{g}_{-1})$ obtained by restricting the adjoint representation to $\mathfrak{g}_{-1}$;

$$\rho = \mathrm{Ad}(\cdot)|_{\mathfrak{g}_{-1}} : G_0 \to \mathrm{GL}(\mathfrak{g}_{-1}), \qquad g_0 \mapsto \mathrm{Ad}(g_0)|_{\mathfrak{g}_{-1}}. \tag{2.2}$$

This representation is infinitesimally effective. The construction then easily implies that ω is a soldering form in the sense of Section 2.1. Hence we obtain a G_0-structure on the manifold M and it turns out that, apart from the case of projective structures, the normality condition on Θ ensures that the Cartan geometry $(\mathcal{G} \to M, \theta)$ is an equivalent encoding of this G_0-structure.

2.4. Weyl structures

In order to work explicitly with a parabolic geometry it is often advantageous to fix a Weyl structure for the parabolic geometry. This gives a description of the Cartan geometry $(\mathcal{G} \to M, \theta)$ in terms of the underlying G_0-structure $(\mathcal{G}_0 \to M, \omega)$ defined above. Here we briefly review the key facts and refer the reader to [5, 6, 7] for details and additional context.

Following [6], we define a Weyl structure for $(\pi : \mathcal{G} \to M, \theta)$ to be a G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the projection $\mathcal{G} \to \mathcal{G}_0$. Writing the Cartan connection θ as $\theta = (\theta_{-1}, \theta_0, \theta_1)$ with θ_i taking values in $\mathfrak{g}_i$, a choice of Weyl

structure σ gives three 1-forms on $\mathcal{G}_0$

$$\begin{aligned} \omega &= \sigma^*\theta_{-1} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1}), \\ \vartheta_\sigma &= \sigma^*\theta_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0), \\ \mathsf{P}^\sigma &= -\psi_B \circ (\sigma^*\theta_1) \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1}^*). \end{aligned} \tag{2.3}$$

Here ω is just the soldering form from Section 2.3 above. The form ϑ_σ is a principal G_0-connection on $\pi_0 : \mathcal{G}_0 \to M$ referred to as the *Weyl connection* determined by σ. For the last component, we let B denotes a suitable constant multiple of the Killing form of $\mathfrak{g}$ and $\psi_B : \mathfrak{g}_1 \to \mathfrak{g}_{-1}^*$ the linear map defined by the rule

$$\psi_B(Z)(X) = B(Z, X)$$

for all $Z \in \mathfrak{g}_1$ and all $X \in \mathfrak{g}_{-1}$. It is well-known that ψ_B is an isomorphism. In the literature on parabolic geometries, ψ_B is usually suppressed from the notation and $\mathfrak{g}_1$ is simply identified with $(\mathfrak{g}_{-1})^*$.

The multiple B of the Killing form is chosen so that the form P^σ represents the so-called *Rho tensor* $\mathbf{P}^\sigma$ of the Weyl structure σ, thought of as a 1-form on M with values in T^*M. Notice that here our convention for the Rho tensor agrees with the classical definitions for conformal an projective structures and hence differs by a sign from [5, 6, 7].

2.5. On the Rho tensor

Here we briefly explain how the normalization condition on the curvature of a $|1|$-graded parabolic geometry leads to a way to explicitly determine the Rho tensor associated to a Weyl structure. We start by introducing a canonical object on M, which will also be useful for other purposes. This comes from the component

$$[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0 \tag{2.4}$$

of the Lie bracket in $\mathfrak{g}$, which is a G_0-equivariant bilinear map. Observe further that the derivative of the representation ρ from (2.2) defines an inclusion $\mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$, which by construction is again induced by the Lie bracket. Together with ψ_B, this shows that (2.4) induces a $\binom{2}{2}$-tensor field on M. We will interpret this below as associating to a vector field $X \in \mathfrak{X}(M)$ and a one-form $\alpha \in \Omega^1(M)$ a $\binom{1}{1}$-tensor field $\{X, \alpha\}$ which in particular can be viewed as a section of $\mathrm{End}(TM, TM)$ or of $\mathrm{End}(T^*M, T^*M)$.

Now given two vector fields $X, Y \in \mathfrak{X}(M)$ and a T^*M-valued one form $\mathbf{P}$, we define a $\binom{1}{3}$-tensor field $\partial\mathbf{P}$ by

$$\partial\mathbf{P}(X, Y) := \{X, \mathbf{P}(Y)\} - \{Y, \mathbf{P}(X)\}. \tag{2.5}$$

By construction, this is skew symmetric in X and Y and thus defines a two-form with values in $\mathrm{End}(TM, TM)$, so this looks like the curvature of a linear connection on TM. Now for a Weyl structure σ, we consider the induced Weyl connection θ_σ. The curvature of this Weyl connection is equivalently encoded by a two-form $\mathbf{R}^\sigma$ with values in $\mathrm{End}(TM, TM)$. Now it turns out that the normalization condition on the Cartan connection implies that the Rho tensor $\mathbf{P}^\sigma$ is uniquely characterized by the fact that $\mathbf{R}^\sigma - \partial\mathbf{P}^\sigma$ has vanishing Ricci type contraction, see [8] or Section

5.2.3 of [6]. Throughout the article we follow the convention of defining the Ricci curvature $\mathrm{Ric}(\nabla)$ of a torsion-free connection ∇ as

$$(X,Y) \mapsto \mathrm{Ric}(\nabla)(X,Y) = \mathrm{tr}\Big(Z \mapsto R^{\nabla}(Z,X)(Y)\Big)$$

where the curvature operator R^{∇} is defined as usual by

$$R^{\nabla}(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Notice that this convention, while common in projective differential geometry, differs (by a swap of the arguments) from the standard convention in Riemannian geometry.

3. The almost para-Kähler structure of a Weyl structure

3.1. Construction of the almost para-Kähler structure

We briefly review the construction of the almost para-Kähler structure associated to a torsion-free $|1|$-graded parabolic geometry given in [5]. Notice that we may think of a torsion-free $|1|$-graded parabolic geometry $(\pi : \mathcal{G} \to M, \theta)$ of type (G, P) on M as a Cartan geometry $(\Pi : \mathcal{G} \to A, \theta)$ of type (G, G_0) on the quotient $A := \mathcal{G}/G_0$. In doing so, the tangent bundle of A becomes $TA = \mathcal{G} \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0)$, where G_0 acts via the adjoint action on $\mathfrak{g}$ and hence on $\mathfrak{g}/\mathfrak{g}_0$. The G_0-module $\mathfrak{g}/\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, where again G_0 acts on both summands via the adjoint representation. Consequently, the tangent bundle of A decomposes into a direct sum of rank n vector bundles $TA = L^+ \oplus L^-$, where $L^{\pm} = \mathcal{G} \times_{G_0} \mathfrak{g}_{\pm 1}$. We consider the 1-form $\eta = \psi_B \circ \theta_1 \in \Omega^1(\mathcal{G}, \mathfrak{g}_{-1}^*)$. Explicitly, we have

$$\eta(v)(X) = B(\theta_1(v), X)$$

for all $v \in T\mathcal{G}$ and all $X \in \mathfrak{g}_{-1}$. It follows from the properties of the Cartan connection and the invariance of B under the adjoint representation that the 1-form η is G_0-equivariant and semibasic for the projection $\Pi : \mathcal{G} \to A$. Consequently, η represents a 1-form $\boldsymbol{\eta}$ on A with values in $(L^-)^* \subset T^*A$. Hence we may view $\boldsymbol{\eta}$ as a section of $T^*A \otimes T^*A$. The symmetric part $h = \mathrm{Sym}\,\boldsymbol{\eta}$ is then a pseudo-Riemannian metric of split-signature on A. This uses that $\psi_B : \mathfrak{g}_1 \to \mathfrak{g}_{-1}^*$ is an isomorphism. The alternating part $\Omega = \mathrm{Alt}\,\boldsymbol{\eta}$ turns out to be a symplectic form by torsion-freeness (see [5, Theorem 3.1]), and the pair (h, Ω) is an almost para-Kähler structure. Furthermore, the sections of $A \to M$ are in bijective correspondence with the Weyl structures for $(\pi : \mathcal{G} \to M, \theta)$. Remarkably, by [5, Theorem 3.5], the normalization conditions for $|1|$-graded parabolic geometries imply that the metric h always is Einstein and hence (h, Ω) is an almost para-Kähler Einstein structure. We refer to [5] for further details and to [2] for recent results about the geometry of 4-dimensional para-Kähler Einstein structures.

3.2. The choice of a Weyl structure

In this section we shall prove the main structural identity (1.1). We start by identifying $\mathcal{G}$ with the product $\mathcal{G}_0 \times \mathfrak{g}_1$ equipped with a suitable right action. An element of $\mathcal{G}_0$ will be denoted by $[u]$, where $u \in \mathcal{G}$. On $\mathcal{G}_0 \times \mathfrak{g}_1$ a P-right action $\hat{R}$ is defined

by the rule

$$([u], Z) \cdot g = ([u \cdot g_0], \mathrm{Ad}(g_0^{-1})(Z) + W)$$

for all $([u], Z) \in \mathcal{G}_0 \times \mathfrak{g}_1$ and all $g = g_0 \exp(W)$ in P and as the basepoint projection we take the map

$$\hat{\pi} : \mathcal{G}_0 \times \mathfrak{g}_1 \to M, \quad ([u], Z) \mapsto \pi_0([u]).$$

With these definitions, $\hat{\pi} : \mathcal{G}_0 \times \mathfrak{g}_1 \to M$ is indeed a right principal P-bundle and moreover, the choice of a Weyl structure identifies this bundle with $\pi : \mathcal{G} \to M$:

Proposition 3.1. *Let $(\pi : \mathcal{G} \to M, \theta)$ be a $|1|$-graded parabolic geometry of type (G, P). Then every Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ induces an isomorphism of principal P-bundles*

$$\Phi_\sigma : \mathcal{G}_0 \times \mathfrak{g}_1 \to \mathcal{G}, \qquad ([u], Z) \mapsto \sigma([u]) \cdot \exp(Z)$$

satisfying

$$(\Phi_\sigma)^* \theta = \mathrm{d}\mathcal{Z} + \sigma^* \theta + [\sigma^* \theta, \mathcal{Z}] + \frac{1}{2}[[\sigma^* \theta, \mathcal{Z}], \mathcal{Z}],$$

where $\mathcal{Z} : \mathcal{G}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ denotes the projection onto the second factor, the brackets are in $\mathfrak{g}$, and we omit writing the pullbacks from the first factor.

For the proof we need the following elementary lemma on the Maurer-Cartan form $\Upsilon_P \in \Omega^1(P, \mathfrak{p})$:

Lemma 3.2. *The exponential map $\exp : \mathfrak{g}_1 \to P$ satisfies $\exp^* \Upsilon_P = \mathrm{d}\,\mathrm{Id}_{\mathfrak{g}_1}$, where $\mathrm{Id}_{\mathfrak{g}_1}$ denotes the identity map on $\mathfrak{g}_1$.*

Proof. For $X, Y \in \mathfrak{g}_1$, we compute

$$\begin{aligned}(\exp^* \Upsilon_P)_X(Y) &= (\Upsilon_P)_{\exp(X)}(\exp'_X(Y)) = (L_{\exp(X)^{-1}})'_{\exp(X)}(\exp'_X(Y)) \\ &= (L_{\exp(X)^{-1}} \circ \exp)'_X(Y) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(-X)\exp(X + tY) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(tY) = Y,\end{aligned}$$

where we use that $[X, X + tY] = 0$ since $[\mathfrak{g}_1, \mathfrak{g}_1] = \{0\}$. □

Proof of Proposition 3.1. First observe that since $[\mathfrak{g}_1, \mathfrak{g}_1] = \{0\}$, we have

$$\exp(Z)\exp(W) = \exp(Z + W)$$

for all $Z, W \in \mathfrak{g}_1$. As a consequence, the standard identity

$$g \exp(X) g^{-1} = \exp(\mathrm{Ad}(g)(X)), \qquad g \in G, X \in \mathfrak{g}$$

implies that for all $g_0 \exp(W) \in P$ and $h_0 \exp(Z) \in P$, we have

$$g_0 \exp(Z) h_0 \exp(W) = g_0 h_0 \exp(\mathrm{Ad}(h_0^{-1})(Z) + W), \tag{3.1}$$

where we use that $\mathrm{Ad}(h_0^{-1})$ preserves $\mathfrak{g}_1$.

Since $\exp : \mathfrak{g}_1 \to P_+$ is a diffeomorphism and $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is equivariant, it easily follows that Φ_σ is a diffeomorphism. In order to verify the equivariancy of

Φ_σ, we compute for all $([u], Z) \in \mathcal{G}_0 \times \mathfrak{g}_1$ and for all $g = g_0 \exp(W) \in P$

$$\begin{aligned}\Phi_\sigma(([u], Z) \cdot g) &= \Phi_\sigma(([u \cdot g_0], \mathrm{Ad}(g_0^{-1})(Z) + W))\\ &= \sigma([u \cdot g_0]) \cdot \exp(\mathrm{Ad}(g_0^{-1})(Z) + W)\\ &= \sigma([u]) \cdot g_0 \exp(\mathrm{Ad}(g_0^{-1})(Z) + W)\\ &= \sigma([u]) \cdot \exp(Z) g_0 \exp(W)\\ &= \sigma([u]) \cdot \exp(Z) g = \Phi_\sigma(([u], Z)) \cdot g,\end{aligned}$$

where we used the definitions of the various mappings as well as (3.1) and the equivariancy of σ. It follows that Φ_σ is a principal P-bundle isomorphism.

For the second part of the lemma we denote by Ad^{-1} the composition of Ad with the inversion in P and compute

$$\begin{aligned}(\Phi_\sigma)^*\theta &= (\sigma, \exp)^*(R^*\theta) = (\sigma, \exp)^*\left(\Upsilon_P + \mathrm{Ad}^{-1} \circ \theta\right)\\ &= \exp^* \Upsilon_P + \sigma^*\left(\mathrm{Ad}^{-1} \circ \theta\right)\end{aligned} \tag{3.2}$$

where we used (2.1) and think of $(\sigma, \exp)$ as a map $\mathcal{G}_0 \times \mathfrak{g}_1 \to \mathcal{G} \times P$. Now for $Z, W \in \mathfrak{g}$ we have the standard identity

$$\mathrm{Ad}(\exp(Z))(W) = \sum_{k=0}^{\infty} \frac{\mathrm{ad}(Z)^k}{k!}(W) = W + [Z, W] + \frac{1}{2}[Z, [Z, W]] + \cdots .$$

As a consequence, we obtain for all $Z \in \mathfrak{g}_1$

$$(R_{\exp(Z)})^*\theta = \mathrm{Ad}(\exp(-Z)) \circ \theta = \theta + [\theta, Z] + \frac{1}{2}[[\theta, Z], Z], \tag{3.3}$$

where we use that the sum terminates after three summands, since $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$. Combining (3.2), (3.3) and Lemma 3.2, we obtain

$$(\Phi_\sigma)^*\theta = \mathrm{d}\mathcal{Z} + \sigma^*\theta + [\sigma^*\theta, \mathcal{Z}] + \frac{1}{2}[[\sigma^*\theta, \mathcal{Z}], \mathcal{Z}], \tag{3.4}$$

as claimed. □

Recall from Section 2.4 that for every choice of Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, $\omega = \sigma^*\theta_{-1} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1})$ is the soldering form of the G_0-structure $\pi_0 : \mathcal{G} \to M$. Moreover, $\vartheta_\sigma = \sigma^*\theta_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ is a principal connection on $\pi_0 : \mathcal{G}_0 \to M$. Using (2.3) and (3.4) we thus obtain

$$(\Phi_\sigma)^*\theta_1 = \mathrm{d}\mathcal{Z} + \sigma^*\theta_1 + [\vartheta_\sigma, \mathcal{Z}] + \frac{1}{2}[[\omega, \mathcal{Z}], \mathcal{Z}]$$

and hence, using (2.3) again, we have

$$(\Phi_\sigma)^*(\psi_B \circ \theta_1) = \psi_B \circ (\mathrm{d}\mathcal{Z} + [\vartheta_\sigma, \mathcal{Z}]) - \mathsf{P}^\sigma + \frac{1}{2}\psi_B \circ ([[\omega, \mathcal{Z}], \mathcal{Z}]) . \tag{3.5}$$

In order to relate this to the Patterson–Walker metric associated to ϑ_σ, we first observe that via ψ_B, the map $\mathcal{Z}$ corresponds to the second projection $\mathcal{G}_0 \times \mathfrak{g}_{-1}^* \to \mathfrak{g}_{-1}^*$. Moreover, in the notation of Section 2.2, the expression $[\vartheta_\sigma, \mathcal{Z}]$ can be written as $(\mathrm{ad} \circ \vartheta_\sigma)(\mathcal{Z})$. Invariance of the bilinear form B implies that for $X \in \mathfrak{g}_{-1}, Y \in \mathfrak{g}_0$ and $Z \in \mathfrak{g}_1$, we get

$$B(\mathrm{ad}(Y)(X), Z) = -B(X, \mathrm{ad}(Y)(Z)).$$

This exactly says that, via ψ_B, the adjoint action on $\mathfrak{g}_1$ corresponds to the dual of the adjoint action on $\mathfrak{g}_{-1}$, which was denoted by ϱ^* in Section 2.2. Together, this shows that the term $\psi_B \circ (\mathrm{d}\mathcal{Z} + [\vartheta_\sigma, \mathcal{Z}])$ exactly gives the $\mathfrak{g}_{-1}^*$-valued 1-form ζ_{ϑ_σ} defined there.

Combining this with (3.5), we obtain

$$(\Phi_\sigma)^*(\psi_B \circ \theta_1) = \zeta_{\vartheta_\sigma} - \mathsf{P}^\sigma + q \tag{3.6}$$

where $q \in \Omega^1(\mathcal{G}_0 \times \mathfrak{g}_{-1}^*, \mathfrak{g}_{-1}^*)$ is given by

$$q = \frac{1}{2}\psi_B \circ \left([[\omega, \psi_B^{-1} \circ \xi], \psi_B^{-1} \circ \xi]\right). \tag{3.7}$$

By construction, q represents a 1-form $\boldsymbol{q}$ on T^*M with values in the pullback of the cotangent bundle of M, which is evidently closely related to the operation on M introduced in Section 2.5. More precisely, for $\alpha \in T^*M$ with $\nu(\alpha) = x$ and a tangent vector $X \in T_\alpha T^*M$, we get

$$\boldsymbol{q}(\alpha)(X) = \tfrac{1}{2}\{\nu'_\alpha(X), \alpha\}(\alpha) \in T_x^*M = (\nu^*T^*M)_\alpha. \tag{3.8}$$

In particular, this shows that $\boldsymbol{q}$ is semibasic for the projection $\nu : T^*M \to M$ and satisfies $(\mathcal{S}_t)^*\boldsymbol{q} = t^2\boldsymbol{q}$, where $\mathcal{S}_t : T^*M \to T^*M$ denotes scaling of a cotangent vector by the factor $t \in \mathbb{R}^*$.

Finally, notice that using ψ_B to identify $\mathfrak{g}_1$ with $\mathfrak{g}_{-1}^*$, we get $T^*M \simeq \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1$. But then the G_0-equivariant diffeomorphism $\Phi_\sigma : \mathcal{G}_0 \times \mathfrak{g}_1 \to \mathcal{G}$ induces a diffeomorphism $\varphi_\sigma : T^*M \to \mathcal{G}/G_0 = A$ and we have

$$(\varphi_\sigma)^*\boldsymbol{\eta} = \boldsymbol{\zeta}_{\vartheta_\sigma} - \nu^*\mathbf{P}^\sigma + \boldsymbol{q}.$$

Recall that $\boldsymbol{\eta}$ is the $(L^-)^*$-valued 1-form on A whose symmetric and alternating part give the almost para-Kähler structure (h, Ω) of the parabolic geometry $(\pi : \mathcal{G} \to M, \theta)$. Recall also that the symmetric part of the form $\boldsymbol{\zeta}_{\vartheta_\sigma}$ is the Patterson–Walker metric h_{ϑ_s} of the Weyl connection ϑ_σ determined by σ. Finally, viewed as a bilinear form on $T_\alpha T^*M$, $\boldsymbol{q}(\alpha)$ turns out to be symmetric. By construction, $\boldsymbol{q}(\alpha)$ is the pullback of a bilinear form on T_xM with $x = \nu(\alpha)$. The latter is induced by the bilinear form on $\mathfrak{g}_{-1}$ that, for some fixed $Z \in \mathfrak{g}_1$, maps (X_1, X_2) to

$$\tfrac{1}{2}B([[X_1, Z], Z], X_2) = -\tfrac{1}{2}B([X_1, Z], [Z, X_2]) = \tfrac{1}{2}B([X_1, Z], [X_2, Z]),$$

so this is obviously symmetric. In summary, we have thus shown:

Theorem 3.3. *Let $(\pi : \mathcal{G} \to M, \theta)$ be a torsion-free $|1|$-graded parabolic geometry with associated almost para-Kähler structure (h, Ω) on A and $\sigma : \mathcal{G}_0 \to \mathcal{G}$ a choice of Weyl structure. Then we have*

$$(\varphi_\sigma)^*h = h_{\vartheta_\sigma} - \nu^*\operatorname{Sym}(\mathbf{P}^\sigma) + \boldsymbol{q},$$

where $\boldsymbol{q}$ is given by formula (3.8).

Remark 3.4 (Local coordinate expression). In terms of a choice of local coordinates $(x^i) : U \to \mathbb{R}^n$ on some open subset $U \subset M$, the metric $(\varphi_\sigma)^*h$ takes the following explicit form. Let $(x^i, \xi_i) : \nu^{-1}(U) \to \mathbb{R}^{2n}$ denote the canonical coordinates induced on $\nu^{-1}(U) \subset T^*M$. The Weyl connection ϑ_σ induces a torsion-free connection ∇ on TM whose Christoffel symbols with respect to the

coordinates (x^i) we denote by Γ^i_{jk}. The Patterson–Walker metric of ϑ_σ can then be expressed as

$$(\mathrm{d}\xi_i - \Gamma^k_{ij}\xi_k \mathrm{d}x^j) \odot \mathrm{d}x^i,$$

where $\odot$ denotes the symmetric tensor product. On $\nu^{-1}(U)$ we thus obtain

$$(\varphi_\sigma)^* h = \Big(\mathrm{d}\xi_i - \Gamma^k_{ij}\xi_k \mathrm{d}x^j - \mathsf{P}_{(ij)} + q_{ij}\Big) \odot \mathrm{d}x^i,$$

where we write $\boldsymbol{q} = q_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ for unique real-valued functions $q_{ij} = q_{ji} : U \to \mathbb{R}$ and $\mathbf{P}^\sigma = \mathsf{P}_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ for unique real-valued functions $\mathsf{P}_{ij} : U \to \mathbb{R}$. Here and henceforth, we employ the summation convention and $\mathsf{P}_{(ij)}$ denotes symmetrization in the indices i, j.

4. Examples

4.1. Projective geometry

Consider an n-dimensional manifold M endowed with a projective structure $[\nabla]$, a class of torsion-free connections that have the same geodesics up to parametrization. This determines a $|1|$-graded parabolic geometry $(\pi : \mathcal{G} \to M, \theta)$, where $G = \mathrm{SL}_\pm(n+1, \mathbb{R})$ is the subgroup of $\mathrm{GL}(n+1, \mathbb{R})$ consisting of matrices whose determinant is ± 1. The grading of its Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R}) = \{B \in \mathfrak{gl}(n+1, \mathbb{R}), \operatorname{tr} B = 0\}$ is given by

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R}^n \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \middle| \, y \in \mathbb{R}^{n*} \right\}$$

and

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} -\operatorname{tr} A & 0 \\ 0 & A \end{pmatrix} \middle| \, A \in \mathfrak{gl}(n, \mathbb{R}) \right\}.$$

Here B is normalised so that

$$B\left(\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \right) = yx = y(x).$$

Next, one computes that for $x \in \mathbb{R}^n$ and $y, z \in \mathbb{R}^{n*}$ we get

$$\left[\left[\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -(yx)z - (zx)y \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

This shows that for $\xi \in T_xM$ and $\alpha \in T^*_xM$ we get $\{\xi, \alpha\}(\alpha) = -2\alpha(\xi)\alpha$. Together with formula (3.8) and the definition of the tautological form τ on T^*M, this shows that

$$\boldsymbol{q} = -\tau \otimes \tau,$$

in agreement with [11].

The Weyl connection $\vartheta_\sigma = \sigma^*\theta_0$ determines a torsion-free connection ∇ on TM which is a representative connection of the projective structure. To compute the associated Rho tensor a similar computation as for formula (4.1) shows that for $X, Z \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$, we get $\{X, \alpha\}(Z) = \alpha(X)Z + \alpha(Z)X$. Using this and formula (2.5) from Section 2.5 we conclude that for $X, Y, Z \in \mathfrak{X}(M)$, we obtain

$$\partial\mathbf{P}^\sigma(X, Y)(Z) = \mathbf{P}^\sigma(Y, X)Z - \mathbf{P}^\sigma(X, Y)Z + \mathbf{P}^\sigma(Y, Z)X - \mathbf{P}^\sigma(X, Z)Y.$$

This easily implies that the Ricci type contraction of $\partial\mathbf{P}^\sigma$ maps X, Y to $n\mathbf{P}^\sigma(X,Y) - \mathbf{P}^\sigma(Y,X)$. Now let $\mathrm{Ric}(\nabla)$ the Ricci type contraction of the curvature of ϑ_σ. Then the discussion in Section 2.5 shows that we must have

$$\mathrm{Ric}(\nabla)(X,Y) = n\mathbf{P}^\sigma(X,Y) - \mathbf{P}^\sigma(Y,X).$$

Symmetrizing and alternating, we conclude that $\mathrm{Sym}\,\mathrm{Ric}(\nabla) = (n-1)\,\mathrm{Sym}\,\mathbf{P}^\sigma$ and $\mathrm{Alt}\,\mathrm{Ric}(\nabla) = (n+1)\,\mathrm{Alt}\,\mathbf{P}^\sigma$, and hence

$$\mathbf{P}^\sigma = \frac{1}{(n-1)}\,\mathrm{Sym}\,\mathrm{Ric}(\nabla) + \frac{1}{(n+1)}\,\mathrm{Alt}\,\mathrm{Ric}(\nabla).$$

Remark 4.1 (Dancing metric). Starting from the standard projective structure on $\mathbb{RP}^2$, the resulting para-Kähler–Einstein structure is defined on $A = \mathrm{SL}(3,\mathbb{R})/\mathrm{GL}(2,\mathbb{R})$. In this case the Einstein metric is referred to as the *dancing metric* [1, 9] because of its significance in the "rolling" of the projective planes $\mathbb{RP}^2$ and $\mathbb{RP}^{2*}$. This para-Kähler–Einstein structure was first constructed in [16] (in any dimension).

Remark 4.2 (para-c-projective compactification). In the projective case, the almost para-Kähler structure on A admits a so-called para-c-projective compactification, see [10], an analog of a c-projective compactification, see [4].

4.2. Conformal geometry

A conformal manifold $(M,[g])$ of dimension $n \geqslant 3$ gives rise to a $|1|$-graded parabolic geometry $(\pi : \mathcal{G} \to M, \theta)$ where G is defined as follows: Consider the matrix

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \mathrm{I}_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

of size $n+2$ and let $G = \mathrm{O}(n+1,1)$ denote the subgroup of $\mathrm{GL}(n+2,\mathbb{R})$ consisting of matrices a satisfying $a^t J a = J$. The Lie algebra $\mathfrak{g} = \mathfrak{o}(n+1,1)$ of G consists of matrices of the form

$$\begin{pmatrix} s & z & 0 \\ x & A & z^t \\ 0 & x^t & -s \end{pmatrix}$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{n*}$ and $A \in \mathfrak{o}(n)$ is a skew-symmetric matrix of size n. The grading of $\mathfrak{g}$ is given by

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x^t & 0 \end{pmatrix} \middle| \, x \in \mathbb{R}^n \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z^t \\ 0 & 0 & 0 \end{pmatrix} \middle| \, y \in \mathbb{R}^n \right\}$$

and

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} s & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -s \end{pmatrix} \middle| \, s \in \mathbb{R}, A \in \mathfrak{o}(n) \right\}.$$

We normalise B such that

$$B\left(\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z^t \\ 0 & 0 & 0 \end{pmatrix} \right) = z(x) = zx.$$

Computing triple brackets as for formula (4.1) one verifies that (with obvious notation) we get for $x, y \in \mathbb{R}^n$ and $z, w \in \mathbb{R}^{n*}$ the expressions

$$[[x, z], w] = -z(x)w - w(x)z + (w^t \cdot z^t)x^t \tag{4.2}$$

$$[[x, z], y] = z(x)y + z(y)x - (x \cdot y)z^t \tag{4.3}$$

Now formula (4.2) shows that for $\xi \in T_x M$ and $\alpha \in T_x^* M$ we get

$$\{\xi, \alpha\}(\alpha) = 2\alpha(\xi)\alpha + g_x^\#(\alpha, \alpha) g_x(\xi, \cdot).$$

Here g is some metric from the conformal class and $g^\#$ is its dual metric, which immediately implies that the operation is conformally invariant. Together with formula (3.8) and the definition of the tautological form τ on T^*M, this shows that

$$\boldsymbol{q} = -\tau \otimes \tau + \frac{1}{2} |\cdot|^2_{g^\sharp} \nu^* g.$$

Here $|\cdot|^2_{g^\sharp}$ is interpreted as a real-valued smooth function on T^*M and the pullback $\nu^* g$ is interpreted as a one-form on T^*M with values in $\nu^* T^* M$.

The Weyl connection $\vartheta_\sigma = \sigma^* \theta_0$ determines a torsion-free connection ∇ on TM which preserves $[g]$ in the sense that for some (any hence any) representative metric $g \in [g]$ there exists a 1-form β such that

$$\nabla g = \beta \otimes g.$$

To compute the Rho-tensor, we first conclude from formula (4.3) that for vector fields $X, Y \in \mathfrak{X}(M)$ and a one-form $\alpha \in \Omega^1(M)$, we get

$$\{X, \alpha\}(Y) = \alpha(X)Y + \alpha(Y)X - g(X, Y) g^\#(\alpha, \cdot).$$

Using this and formula (2.5) from Section 2.5 we conclude that for $X, Y, Z \in \mathfrak{X}(M)$, we can write $\partial \mathbf{P}^\sigma(X, Y)(Z)$ as

$$\begin{aligned} &\mathbf{P}^\sigma(Y, X)Z + \mathbf{P}^\sigma(Y, Z)X - g(X, Z) g^\#(\mathbf{P}^\sigma(Y)) \\ &-\mathbf{P}^\sigma(X, Y)Z - \mathbf{P}^\sigma(X, Z)Y + g(Y, Z) g^\#(\mathbf{P}^\sigma(X)). \end{aligned}$$

To form the Ricci-type contraction of this, we have to take a local orthonormal frame, insert each element for X and then take the inner product with the same element and sum the results. This sends (Y, Z) to

$$(n-1)\mathbf{P}^\sigma(Y, Z) - \mathbf{P}^\sigma(Z, Y) + g(Y, Z) \operatorname{tr}_{g^\#}(\mathbf{P}^\sigma).$$

Observe that in the literature on conformal geometry usually only the case of Levi-Civita connections of metrics in the conformal class is discussed, for which $\mathbf{P}^\sigma$ is automatically symmetric. Anyway, the discussion in Section 2.5 shows that the above expression has to coincide with $\operatorname{Ric}(\nabla)$. To conclude the discussion as in Section 4.1 above, we now have to compute the alternation, and the trace-free part and the trace part of the symmetrization, which gives

$$\begin{aligned} \operatorname{Sym}_0 \operatorname{Ric}(\nabla) &= (n-2) \operatorname{Sym}_0 \mathbf{P}^\sigma \\ \operatorname{Alt} \operatorname{Ric}(\nabla) &= n \operatorname{Alt} \mathbf{P}^\sigma \\ g \operatorname{tr}_{g^\#}(\operatorname{Ric}(\nabla)) &= (2n-2) g \operatorname{tr}_{g^\#}(\mathbf{P}^\sigma). \end{aligned}$$

Observe that for a Levi-Civita connection $\operatorname{tr}_{g^\#}(\operatorname{Ric}(\nabla))$ is the scalar curvature of the metric g. In any case, we immediately get the general formula

$$\mathbf{P}^\sigma = \frac{1}{n-2}\operatorname{Sym}_0 \operatorname{Ric}(\nabla) + \frac{1}{n}\operatorname{Alt}\operatorname{Ric}(\nabla) + \frac{1}{n(n-2)} g \operatorname{tr}_{g^\#}(\operatorname{Ric}(\nabla)).$$

4.3. Grassmannian geometry

An *almost Grassmannian structure of type* (m, n) on a manifold M consists of two real vector bundles E and F on M, of rank m and n and vector bundle isomorphisms $TM \simeq E^* \otimes F \simeq \operatorname{Hom}(E, F)$ and $\Lambda^m E^* \cong \Lambda^n F$. Here E^* denotes the dual of E and the isomorphism between the top exterior powers will not be relevant for us. An almost Grassmannian structure on M gives rise to a $|1|$-graded parabolic geometry $(\pi : \mathcal{G} \to M, \theta)$ where $G = \mathrm{SL}(n+m, \mathbb{R})$. The structure is called *Grassmannian* if it admits a compatible torsion-free connection on TM, which is equivalent to torsion-freeness of the parabolic geometry. The grading of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+m, \mathbb{R})$ of G is given by

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \middle| \, x \in M(n \times m, \mathbb{R}) \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \middle| \, y \in M(m \times n, \mathbb{R}) \right\}$$

and

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \middle| \, A \in \mathfrak{gl}(n, \mathbb{R}), B \in \mathfrak{gl}(m, \mathbb{R}), \operatorname{tr}(A) + \operatorname{tr}(B) = 0 \right\},$$

where $M(n \times m, \mathbb{R})$ denotes the vector space of $(n \times m)$-matrices with real entries. The main case of interest for our purpose is $m = 2$, $n \geq 2$, for which there are examples of such geometries that are torsion-free but not locally isomorphic to G/P. For $m = n = 2$ such a structure is equivalent to a conformal structure of neutral signature. For $m, n > 2$, any torsion-free structure is locally isomorphic to G/P, but our results still are of interest, since there is the freedom in the choice of Weyl structure.

As the invariant form B we use the trace form, which leads to

$$B\left(\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \right) = \operatorname{tr}(zx) = \operatorname{tr}(xz).$$

Formally, the setup looks very similar to projective structures (which correspond to the case $m = 1$). This is also reflected in the structure of the triple brackets, which formally look very similar to (4.1): For $x, y \in M(n \times m, \mathbb{R})$ and $z, w \in M(m \times n, \mathbb{R})$ we get (in obvious notation)

$$[[x, z], w] = -zxw - wxz \qquad [[x, z], y] = xzy + yzx. \tag{4.4}$$

However, here we have matrix multiplications so for example xz is a 2×2-matrix and zxw is not simply a multiple of w.

The easiest way to encode our operations geometrically is to define one additional operation on a manifold M endowed with an almost Grassmannian structure. Since $TM \cong E^* \otimes F$, we get $T^*M \cong E \otimes F^* \cong \operatorname{Hom}(F, E)$ and thus composition of linear maps induces a bilinear bundle map $T^*M \times TM \to \operatorname{End}(E, E)$, which we denote by $(\alpha, X) \mapsto \alpha \circ X$ both on elements and on sections. This can be viewed as an "refinement" of the dual pairing, since by definition we get $\alpha(X) = \operatorname{tr}(\alpha \circ X)$, where tr denotes the point-wise trace. There clearly are analogous composition operations $TM \times \operatorname{End}(E, E) \to TM$ and $\operatorname{End}(E, E) \times T^*M \to T^*M$ (and others

that we don't need here). In this language, the first formula in (4.4) readily shows that for $\alpha \in T^*_x M$ and $\xi \in T_x M$, we get $\{\xi, \alpha\}(\alpha) = -2(\alpha \circ \xi) \circ \alpha$.

Next, we get a corresponding refinement $\tau^G \in \Omega^1(T^*M, \mathrm{End}(\nu^*E, \nu^*E))$ of the tautological one-form τ on T^*M. By definition, for $\alpha \in T^*M$ with $\nu(\alpha) = x \in M$, the fiber of $\mathrm{End}(\nu^*E, \nu^*E)$ over α equals $\mathrm{End}(E_x, E_x)$, so for $\xi \in T_\alpha T^*M$, we can define

$$\tau^G(\alpha)(\xi) := \alpha \circ \nu'_\alpha(\xi).$$

By definition, the tautological form τ is then recovered as $\tau = \mathrm{tr}(\tau^G)$, where again tr denotes a point-wise trace in the values of the form. Using formula (3.8) we readily conclude that for $\alpha \in T^*M$ and $\xi, \eta \in T_\alpha T^*M$ we get

$$\boldsymbol{q}(\alpha)(\xi, \eta) = -\mathrm{tr}((\alpha \circ \nu'_\alpha(\xi)) \circ (\alpha \circ \nu'_\alpha(\eta))),$$

which is evidently symmetric in ξ and η. To connect more closely to the other cases, we can write this as $\boldsymbol{q} = -\mathrm{tr}(\tau^G \otimes \tau^G)$, where we agree that the tensor product of one-forms with values in $\mathrm{End}(\nu^*E, \nu^*E)$ includes a composition of the values, i.e. $(A \otimes B)(\xi, \eta) = A(\xi) \circ B(\eta)$.

The description of the Rho tensor is also similar to the projective case, with some complications caused by the matrix multiplications.The Weyl connection $\vartheta_\sigma = \sigma^*\theta_0$ determines a torsion-free connection ∇ on TM which is induced by connections on E and F that are compatible with the isomorphism of the top exterior powers. Now the second equation in (4.4) readily shows that for $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$ we get $\{\alpha, X\}(Y) = X \circ (\alpha \circ Y) + Y \circ (\alpha \circ X)$. Using this and formula (2.5) from Section 2.5 we conclude that for $X, Y, Z \in \mathfrak{X}(M)$, $\partial\mathbf{P}^\sigma(X, Y)(Z)$ is given by

$$X \circ (\mathbf{P}^\sigma(Y) \circ Z) + Z \circ (\mathbf{P}^\sigma(Y) \circ X) - Y \circ (\mathbf{P}^\sigma(X) \circ Z) - Z \circ (\mathbf{P}^\sigma(X) \circ Y)$$

To compute the action of the Ricci type contraction on Y and Z, we have to insert the elements of a basis for X and contract (i.e. take the trace of the composition) with the dual basis element and sum up over the basis. To write up the result, we need additional notation. We have to view $\mathbf{P}^\sigma(x)$ as an element in $\otimes^2 T^*_x M$ and identifying $T_x M$ with $\mathrm{Hom}(F_x, E_x)$ such an element defines a linear map $F_x \otimes F_x \to E_x \otimes E_x$. But for such a map, we can separately swap the input or the output and hence independently symmetrize or alternate in the input and the output. Writing t_E for the twist map on $E \otimes E$ and t_F for the twist map on $F \otimes F$, one verifies that the Ricci type contraction of $\partial\mathbf{P}^\sigma$ can be written as

$$(mn + 1)\mathbf{P}^\sigma - t_E \circ \mathbf{P}^\sigma - \mathbf{P}^\sigma \circ t_F.$$

Observe that this is consistent with the result in the projective case, since for $m = 1$, we have $t_E = \mathrm{id}$. By the discussion in Section 2.5, this again has to coincide with the Ricci type contraction $\mathrm{Ric}(\nabla)$ of the curvature of ∇. Now we can compose this equation on both sides with either a symmetrization or an alternation to obtain

$$\begin{aligned}
\mathrm{Sym} \circ \mathrm{Ric}(\nabla) \circ \mathrm{Sym} &= (mn - 1)\, \mathrm{Sym} \circ \mathbf{P}^\sigma \circ \mathrm{Sym} \\
\mathrm{Alt} \circ \mathrm{Ric}(\nabla) \circ \mathrm{Alt} &= (mn + 3)\, \mathrm{Alt} \circ \mathbf{P}^\sigma \circ \mathrm{Alt} \\
\mathrm{Sym} \circ \mathrm{Ric}(\nabla) \circ \mathrm{Alt} &= (mn + 1)\, \mathrm{Sym} \circ \mathbf{P}^\sigma \circ \mathrm{Alt} \\
\mathrm{Alt} \circ \mathrm{Ric}(\nabla) \circ \mathrm{Sym} &= (mn + 1)\, \mathrm{Alt} \circ \mathbf{P}^\sigma \circ \mathrm{Sym}\,.
\end{aligned}$$

From this, one deduces an explicit formula for $\mathbf{P}^\sigma$ as before.

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