created on 2025/04/29 at 15:21:42
arXiv version: https://arxiv.org/abs/1908.10325
journal version: Commun. Contemp. Math. 25 (2023), Article No. 2250026, 40 pp.

Geometric theory of Weyl structures

ANDREAS ČAP AND THOMAS METTLER

ABSTRACT. Given a parabolic geometry on a smooth manifold M, we study a natural affine bundle $A \to M$, whose smooth sections can be identified with Weyl structures for the geometry. We show that the initial parabolic geometry defines a reductive Cartan geometry on A, which induces an almost bi-Lagrangian structure on A and a compatible linear connection on TA. We prove that the split-signature metric given by the almost bi-Lagrangian structure is Einstein with non-zero scalar curvature, provided the parabolic geometry is torsion-free and $|1|$-graded. We proceed to study Weyl structures via the submanifold geometry of the image of the corresponding section in A. For Weyl structures satisfying appropriate non-degeneracy conditions, we derive a universal formula for the second fundamental form of this image. We also show that for locally flat projective structures, this has close relations to solutions of a projectively invariant Monge-Ampere equation and thus to properly convex projective structures.

1. Introduction

Parabolic geometries form a class of geometric structures that look very diverse in their standard description. This class contains important and well-studied examples like conformal and projective structures, non-degenerate CR structures of hypersurface type, path geometries, quaternionic contact structures, and various types of generic distributions. They admit a uniform conceptual description as Cartan geometries of type (G, P) for a semisimple Lie group G and a parabolic subgroup $P \subset G$ in the sense of representation theory. Such a geometry on a smooth manifold M is given by a principal P-bundle $p : \mathcal{G} \to M$ together with a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which defines an equivariant trivialization of the tangent bundle $T\mathcal{G}$. A standard reference for parabolic geometries is [16].

The group P can be naturally written as a semi-direct product $G_0 \ltimes P_+$ of a reductive subgroup G_0 and a nilpotent normal subgroup P_+. For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ the quotient $\mathcal{G}_0 := \mathcal{G}/P_+ \to M$ is a principal G_0-bundle, and some parts of ω can be descended to that bundle. In the simplest cases, this defines a usual first order G_0-structure on M, in more general situations a filtered analog of such a structure. Thus the Cartan geometry can be viewed as an extension of a first order structure. This reflects the fact that morphisms of parabolic geometries are in general not

Date: February 2, 2022.

determined locally around a point by their 1-jet in that point, and the Cartan connection captures the necessary higher order information.

To work explicitly with parabolic geometries, one often chooses a more restrictive structure, say a metric in a conformal class, a connection in a projective class or a pseudo-Hermitian structure on a CR manifold, expresses things in terms of this choice and studies the effect of different choices. It turns out that there is a uniform way to do this that can be applied to all parabolic geometries, namely the concept of Weyl structures introduced in [15], see Chapter 5 of [16] for an improved exposition. Choosing a Weyl structure, one in particular obtains a linear connection on any natural vector bundle associated to a parabolic geometry, as well as an identification of higher order geometric objects like tractor bundles with more traditional natural bundles. The set of Weyl structures always forms an affine space modeled on the space on one-forms on the underlying manifold, and there are explicit formulae for how a change of Weyl structure affects the various derived quantities.

The initial motivation for this article were the results in [19] on projective structures. Such a structure on a smooth manifold M is given by an equivalence class $[\nabla]$ of torsion-free connections on its tangent bundle, where two connections are called equivalent if they have the same geodesics up to parametrization. While these admit an equivalent description as a parabolic geometry, the underlying structure $\mathcal{G}_0^{\mathcal{C}} \to M$ is the full frame bundle of M and thus contains no information. Hence there is the natural question, whether a projective structure can be encoded into a first order structure on some larger space constructed from M. Indeed, in [19], the authors associate to a projective structure on an n-dimensional manifold M a certain rank n affine bundle $A \to M$, whose total space can be canonically endowed with a neutral signature metric h, as well as a non-degenerate 2-form Ω. It turns out that the metric h is Einstein and Ω is closed. Moreover, the pair (h, Ω) is related by an endomorphism of TA which squares to the identity map and its eigenbundles $L^{\pm}$ are Lagrangian with respect to Ω. Equivalently, we may think of the pair (h, Ω) as an *almost para-Kähler structure* or as an *almost bi-Lagrangian structure* (Ω, L^+, L^-) on A, see Section 3.1 for the formal definition and more details.

In addition, it is observed that the sections of $A \to M$ are in bijective correspondence with the connections in the projective class. Consequently, all the submanifold notions of symplectic – and pseudo-Riemannian geometry can be applied to the representative connections of $[\nabla]$. This leads in particular to the notion of a minimal Lagrangian connection [28]. As detailed below, this concept has close relations to the concept of properly convex projective structures. These in turn provide a connection to the study of representation varieties and higher Teichmüller spaces, see [32] for a survey.

In an attempt to generalize these constructions to a larger class of parabolic geometries, we were led to a definition of $A \to M$ that directly leads to an interpretation as a bundle of Weyl structures. This means that the space of sections of $A \to M$ can be naturally identified with the space of Weyl

structures for the geometry $(p : \mathcal{G} \to M, \omega)$. At some stage it was brought to our attention that a bundle of Weyl structures had been defined in that way already in the article [21] by M. Herzlich in the setting of general parabolic geometries. In this article, Herzlich gave a rather intricate argument for the existence of a connection on TA and used this to study canonical curves in parabolic geometries.

The crucial starting point for our results here is that a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ can also naturally be interpreted as a Cartan geometry on A with structure group G_0. This immediately implies that for any type of parabolic geometry, there is a canonical linear connection on any natural vector bundle over A as well as natural almost bi-Lagrangian structure on A that is compatible with the canonical connection. So in particular, we always obtain a non-degenerate two-form $\Omega \in \Omega^2(A)$, a neutral signature metric h on TA as well as a decomposition $TA = L^- \oplus L^+$ as a sum of Lagrangian subbundles.

Using the interpretation via Cartan geometries, it turns out that all elements of the theory of Weyl structures admit a natural geometric interpretation in terms of pulling back operations on A via the sections defined by a Weyl structure. This works for general parabolic geometries as shown in Section 2. In particular, we show that Weyl connections are obtained by pulling back the canonical connection on A, while the Rho tensor (or generalized Schouten tensor) associated to a Weyl connection is given by the pullback of a canonical L^+-valued one-form on A.

We believe that this interpretation of Weyl structures should be a very useful addition to the tool set available for the study of parabolic geometries. Indeed, working with the canonical geometric structures on A compares to the standard way of using Weyl structures, like working on a frame bundle compares to working in local frames.

For the second part of the article, we adopt a different point of view. From Section 3 on, we use the relation to Weyl structures as a tool for the study of the intrinsic geometric structure on A and its relation to non-linear invariant PDE. Our first main result shows that one has to substantially restrict the class of geometries in order to avoid getting into exotic territory. Recall that for a parabolic subgroup $P \subset G$ the corresponding Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ can be realized as the non-negative part in a grading $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ of $\mathfrak{g}$, which is usually called a $|k|$-grading. There is a subclass of parabolic geometries that is often referred to as *AHS structures*, see e.g. [3, 4, 14], which is the case $k = 1$, see Section 2.2 and Theorem 3.2 for more details. This is exactly the case in which the underlying structure $\mathcal{G}_0 \to M$ is an ordinary first order G_0-structure. In particular, there is the notion of intrinsic torsion for this underlying structure. Vanishing of the intrinsic torsion is equivalent to the existence of a torsion free connection compatible with the structure and turns out to be equivalent to torsion-freeness of the Cartan geometry $(p : \mathcal{G} \to M, \omega)$. Using this background, we can formulate the first main result of Section 3, that we prove as Theorem 3.1:

Theorem. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and $\pi : A \to M$ its associated bundle of Weyl structures. Then the natural 2-form $\Omega \in \Omega^2(A)$ is closed if and only if (G, P) corresponds to a $|1|$-grading and the Cartan geometry $(p : \mathcal{G} \to M, \omega)$ is torsion-free.*

Hence we restrict our considerations to torsion-free AHS structures from this point on. Apart from projective and conformal structures, this contains also Grassmannian structures of type $(2, n)$ and quaternionic structures, for which there are many non-flat examples. For several other AHS structures, torsion-freeness implies local flatness, but the locally flat case is of particular interest for us anyway. Our next main result, which we prove in Theorem 3.5, vastly generalizes [**19**]:

Theorem. *For any torsion-free AHS structure, the pseudo-Riemannian metric h induced by the canonical almost bi-Lagrangian structure on the bundle A of Weyl structures is an Einstein metric with non-zero scalar curvature.*

While one could prove the aforementioned Theorems on a case by case basis by using the techniques from [**19**], our arguments instead rely on a careful analysis of the properties of the curvature tensor of the induced connection on TA. Following [**28**], we next initiate the study of Weyl structures via the geometry of submanifolds in A. We call a Weyl structure $s : M \to A$ of a torsion-free AHS structure *Lagrangian* if $s : M \to (A, \Omega)$ is a Lagrangian submanifold. Likewise, s is called *non-degenerate* if $s : M \to (A, h)$ is a non-degenerate submanifold. We show that a Weyl structure is Lagrangian if and only if its Rho tensor is symmetric and that it is non-degenerate if and only if the symmetric part of its Rho tensor is non-degenerate.

In Theorem 3.12 we characterize Lagrangian Weyl structures that lead to totally geodesic submanifolds $s(M) \subset A$, which provides a connection to Einstein metrics and reductions of projective holonomy. If s in addition is non-degenerate, then there is a well defined second fundamental form of $s(M)$ with respect to any linear connection on TA that is metric for h and we show that this admits a natural interpretation as a $\binom{1}{2}$-tensor field on M. In our next main result, Theorem 3.13, we give explicit formulae for the second fundamental forms of the canonical connection and the Levi-Civita connection of h. These are universal formulae in terms of the Weyl connection, the Rho-tensor, and its inverse, which are valid for all torsion-free AHS structures. As an application, we are able to characterize non-degenerate Lagrangian Weyl structures that are minimal submanifolds in (A, h) in terms of a universal PDE. Again, this is a vast generalization of [**28, Theorem 4.4**], where merely the case of projective structures on surfaces was considered.

In Section 4 we connect our results to the study of fully non-linear invariant PDE on AHS structures. A motivating example arises from the work of E. Calabi. In [**7**], Calabi related complete affine hyperspheres to solutions of a certain Monge-Ampère equation. This Monge-Ampère equation, when interpreted correctly, is an invariant PDE that one can associate to a projective structure and it is closely linked to properly convex projective manifolds,

see [**26, Theorem 4**]. In Theorem 4.4, we relate Calabi's equation to our equation for a minimal Lagrangian Weyl structure and as Theorem 4.6, we obtain:

Corollary. *Let $(M, [\nabla])$ be a closed oriented locally flat projective manifold. Then $[\nabla]$ is properly convex if and only if $[\nabla]$ arises from a minimal Lagrangian Weyl structure whose Rho tensor is positive definite.*

The convention for the Rho tensor used here is chosen to be consistent with [**16**]. This convention is natural from a Lie theoretic viewpoint, but differs from the standard definition in projective – and conformal differential geometry by a sign. It should also be noted that a relation between properly convex projective manifolds and minimal Lagrangian submanifolds has been observed previously in [**23, 22**] (but not in the context of Weyl structures).

The notion of convexity for projective structures is only defined for locally flat structures. The above Corollary thus provides a way to generalize the notion of a properly convex projective structure to a class of projective structures that are possibly curved, namely those arising from a minimal Lagrangian Weyl structure. Going beyond projective geometry, this class of differential geometric structures is well-defined for all torsion-free AHS structures.

We conclude the article by showing that there are analogs of the projective Monge-Ampère equation for other AHS structures, and that these always can be described in terms of the Rho tensor, which provides a relation to submanifold geometry of Weyl structures. These topics will be studied in detail elsewhere.

Acknowledgments

AČ gratefully acknowledges support by the Austrian Science Fund (FWF): P 33559-N. A part of the research for this article was carried out while TM was visiting Forschungsinstitut für Mathematik (FIM) at ETH Zürich. TM thanks FIM for its hospitality and DFG for partial funding through the priority programme SPP 2026 "Geometry at Infinity". We also thank M. Dunajski, J. Šilhan and V. Žadnìk for helpful discussions and the anonymous referee for interesting comments that helped improving the article significantly.

2. The bundle of Weyl structures

This section works in the setting of general parabolic geometries. We assume that the reader is familiar with the the basic concepts and only briefly collect what we need about parabolic geometries and Weyl structures. Then we define the bundle of Weyl structures and identify some of the geometric structures that are naturally induced on its total space. We then prove existence of a canonical connection and explain how these structures can be used as an equivalent encoding of the theory of Weyl structures.

2.1. Parabolic geometries

The basic ingredient needed to specify a type of parabolic geometry is a semisimple Lie algebra $\mathfrak{g}$ that is endowed with a so-called $|k|$-grading. This is a decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

of $\mathfrak{g}$ into a direct sum of linear subspaces such that

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_\ell = \{0\}$ for $|\ell| > k$.
- No simple ideal of $\mathfrak{g}$ is contained in the subalgebra $\mathfrak{g}_0$.
- The subalgebra $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is generated by $\mathfrak{g}_1$.

In particular, this implies that the Lie subalgebra $\mathfrak{g}_0$ naturally acts on each of the spaces $\mathfrak{g}_i$ via the restriction of the adjoint action. Moreover, $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is a Lie subalgebra of $\mathfrak{g}$, which turns out to be a parabolic subalgebra in the sense of representation theory.

Such $|k|$-gradings can be easily described in terms of the structure theory of semisimple Lie algebras, see Section 3.2 of [16]. In particular, it turns out that any parabolic subalgebra is obtained in this way and, essentially, the classification of gradings is equivalent to the classification of parabolic subalgebras. Further, the decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is the reductive Levi decomposition, so it is a semi-direct product, $\mathfrak{p}_+$ is the nilradical of $\mathfrak{p}$, and the subalgebra $\mathfrak{g}_0$ is reductive. Of course, also $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is a Lie subalgebra of $\mathfrak{g}$, which is nilpotent by the grading property. It turns out that $\mathfrak{g}_-$ and $\mathfrak{p}_+$ are isomorphic.

Next, one chooses a Lie group G with Lie algebra $\mathfrak{g}$. Then the normalizer of $\mathfrak{p}$ in G has Lie algebra $\mathfrak{p}$, and one chooses a closed subgroup $P \subset G$ lying between this normalizer and its connected component of the identity. The subgroup P naturally acts on $\mathfrak{g}$ and $\mathfrak{p}$ via the adjoint action. More generally, one puts $\mathfrak{g}^i := \oplus_{j \geq i} \mathfrak{g}_j$ to define a filtration of $\mathfrak{g}$ by linear subspaces that is invariant under the adjoint action of P. This makes $\mathfrak{g}$ into a filtered Lie algebra in the sense that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$.

Having made these choices, there is the concept of a *parabolic geometry of type* (G, P) on a manifold M of dimension $\dim(G/P)$. This is defined as a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), which means that $p : \mathcal{G} \to M$ is a principal P-bundle and that $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a *Cartan connection*. This in turn means that ω is equivariant for the principal right action, so $(r^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$, reproduces the generators of fundamental vector fields, and that $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$. In addition, one requires two conditions on the curvature of ω, which are called *regularity* and *normality*, which we don't describe in detail.

While Cartan geometries provide a nice uniform description of parabolic geometries, this should be viewed as the result of a theorem rather than a definition. To proceed towards more common descriptions of the geometries, one first observes that the Lie group P can be decomposed as a semi-direct product. On the one hand, the exponential map restricts to a diffeomorphism from $\mathfrak{p}_+$ onto a closed normal subgroup $P_+ \subset P$. On the other hand, one defines a closed subgroup $G_0 \subset P$ as consisting of those elements, whose

adjoint action preserves the grading of $\mathfrak{g}$ and observes that this has Lie algebra $\mathfrak{g}_0$. Then the inclusion of G_0 into P induces an isomorphism $G_0 \to P/P_+$.

Using this, one can pass from the Cartan geometry $(p : \mathcal{G} \to M, \omega)$ to an underlying structure by first forming the quotient $\mathcal{G}_0 := \mathcal{G}/P_+$, which is a principal G_0-bundle. Moreover, for each $i = -k, \dots, k$, there is a smooth subbundle $T^i\mathcal{G} \subset T\mathcal{G}$ consisting of those tangent vectors that are mapped to $\mathfrak{g}^i \subset \mathfrak{g}$ by ω. Since $T^1\mathcal{G}$ is the vertical bundle of $\mathcal{G} \to \mathcal{G}_0$, these subbundles descend to a filtration $\{T^i\mathcal{G}_0 : i = -k, \dots, 0\}$ of $T\mathcal{G}_0$. Moreover, for each $i < 0$, the component of ω in $\mathfrak{g}_i$ descends define a smooth section of the bundle $L(T^i\mathcal{G}_0, \mathfrak{g}_i)$ of linear maps, so this can be viewed as a partially defined $\mathfrak{g}_i$-valued differential form.

The simplest case here is $k = 1$, for which the geometries in question are often referred to as *AHS structures*. In this case, one obtains a $\mathfrak{g}_{-1}$-valued one-form θ on $\mathcal{G}_0$, which is G_0-equivariant and whose kernel in each point is the vertical subbundle. This means that $(p_0 : \mathcal{G}_0 \to M, \theta)$ in this case simply is a classical first order structure corresponding to the adjoint action of G_0 on $\mathfrak{g}_{-1}$ (which turns out to be infinitesimally effective). According to a result of Kobayashi and Nagano (see [24]), the resulting class of structures for simple $\mathfrak{g}$ is very peculiar, since these are the only irreducible first order structures of finite type, for which the first prolongation is non-trivial. This class contains important examples, like conformal structures, almost quaternionic structures, and almost Grassmannian structures.

For general k, there is an interpretation of $\mathcal{G}_0$ and the partially defined forms as a filtered analogue of a first order structure. This involves a filtration of the tangent bundle TM by smooth subbundles T^iM for $i = -k, \dots, -1$ with prescribed (non–)integrability properties together with a reduction of structure group of the associated graded vector bundle to the tangent bundle. This leads to examples like hypersurface-type CR structures, in which the filtration is equivalent to a contact structure, while the reduction of structure group is defined by an almost complex structure on the contact subbundle. Further important example of such structures are path geometries, quaternionic contact structures and various types of generic distributions.

Except for two cases, the Cartan geometry can be uniquely (up to isomorphism) recovered from the underlying structure (see Section 3.1 of [16]), and indeed this defines an equivalence of categories. So in this case, one has two equivalent descriptions of the structure. The two exceptional cases are projective structures and a contact analogue of those. In these cases, the underlying structure contains no information respectively describes only the contact structure, and one in addition has to choose an equivalence class of connections in order to describe the structure. Still, these fit into the general picture with respect to Weyl structures, which we discuss next.

2.2. Weyl structures

These provide the basic tool to explicitly translate between the description of a parabolic geometry as a Cartan geometry and the picture of the underlying structure. So let us suppose that $(p : \mathcal{G} \to M, \omega)$ is a Cartan geometry of

type (G,P) and that $p_0 : \mathcal{G}_0 \to M$ is the underlying structure described in Section 2.1. The original definition of a Weyl structure used in [15] is as a G_0-equivariant section σ of the natural projection $q : \mathcal{G} \to \mathcal{G}_0 = \mathcal{G}/P_+$. One shows that such sections always exist globally and by definition, they provide reductions of the principal P-bundle $p : \mathcal{G} \to M$ to the structure group $G_0 \subset P$. As a representation of G_0, the Lie algebra $\mathfrak{g}$ splits as $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ (and indeed further according to the $|k|$-grading). Thus, the pullback $\sigma^*\omega$ splits accordingly into a sum of three G_0-equivariant one-forms with values in $\mathfrak{g}_-$, $\mathfrak{g}_0$ and $\mathfrak{p}_+$, respectively, which then admit nice interpretations in terms of the underlying structure. The $\mathfrak{g}_0$-component defines a principal connection on $\mathcal{G}_0$, which induces the *Weyl connections* on associated bundles. The component in $\mathfrak{p}_+$ descends to a one-form on M with values in the associated graded to the cotangent bundle T^*M, which is the *Rho-tensor* associated to the Weyl structure. The $\mathfrak{g}_-$-component also descends to M and provides an isomorphism between the tangent bundle TM and its associated graded bundle. For the structures we consider in this article, this component coincides with the soldering form that identifies $\mathcal{G}_0$ as a reduction of structure group of TM.

As observed in [21], *any* reduction of $p : \mathcal{G} \to M$ to the structure group $G_0 \subset P$ comes from a Weyl structure. This is because the composition of q with the principal bundle morphism defining such a reduction clearly is an isomorphism of G_0-principal bundles. Thus one could equivalently define a Weyl structure as such a reduction of structure group and then observe that this defines a G_0-equivariant section of $q : \mathcal{G} \to \mathcal{G}_0$. It is a classical result that reductions of $\mathcal{G}$ to the structure group G_0 can be equivalently described as smooth sections of the associated bundle with fiber P/G_0. This motivates the following definition from [21].

Definition 2.1. The *bundle of Weyl structures* associated to the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is $\pi : A := \mathcal{G} \times_P (P/G_0) \to M$.

The correspondence between Weyl structures and smooth sections of $\pi : A \to M$ can be easily made explicit. Given a G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ one considers the map sending $u_0 \in \mathcal{G}_0$ to the class of $(\sigma(u_0), eG_0)$ in $\mathcal{G} \times_P (P/G_0)$, where $e \in P$ is the neutral element. By construction, the resulting smooth map $\mathcal{G}_0 \to A$ is constant on the fibers of $p_0 : \mathcal{G}_0 \to M$ and thus descends to a smooth map $s : M \to A$, which is a section of π by construction. Conversely, a section s of π corresponds to a smooth, G_0-equivariant map $f : \mathcal{G} \to P/G_0$ characterized by the fact that $s(x)$ is the class of $(u, f(u))$ for each u in the fiber of $\mathcal{G}$ over x. But then $f^{-1}(eG_0)$ is a smooth submanifold of $\mathcal{G}$ on which the projection $q : \mathcal{G} \to \mathcal{G}_0$ restricts to a G_0-equivariant diffeomorphism. The inverse of this diffeomorphism gives the Weyl structure determined by s.

From the definition, we can verify that the bundle of Weyl structures is similar to an affine bundle. This will also provide the well known affine structure on Weyl structures in our picture. To formulate this, recall first that the parabolic subgroup $P \subset G$ is a semi-direct product of the subgroup $G_0 \subset P$ and the normal subgroup $P_+ \subset P$. In particular, any element

$g \in P$ can be uniquely written as $g_0 g_1$ with $g_0 \in G_0$ and $g_1 \in P_+$, compare with Theorem 3.1.3 of [16], and of course $g_0 g_1 = (g_0 g_1 g_0^{-1}) g_0$ provides the corresponding decomposition in the opposite order.

Proposition 2.2. *Let $\pi : A \to M$ be the bundle of Weyl structures associated to a parabolic geometry $(p : \mathcal{G} \to M, \omega)$. Then sections of $\pi : A \to M$ can be naturally identified with smooth functions $f : \mathcal{G} \to P_+$ such that $f(u \cdot (g_0 g_1)) = g_1^{-1} g_0^{-1} f(u) g_0$ for each $u \in \mathcal{G}$, $g_0 \in G_0$ and $g_1 \in P_+$.*

Fixing one function f that satisfies this equivariancy condition, any other function which is equivariant in the same way can be written as $\hat{f}(u) = f(u)h(u)$, where $h : \mathcal{G} \to P_+$ is a smooth function such that $h(u \cdot (g_0 g_1)) = g_0^{-1} h(u) g_0$.

Proof. The inclusion $P_+ \hookrightarrow P$ induces a smooth map $P_+ \to P/G_0$, and from the decomposition of elements of P described above, we readily see that this is surjective. On the other hand, writing the quotient projection $P \to G_0$ as α, the map $g \mapsto g\alpha(g)^{-1}$ induces a smooth inverse, so P/G_0 is diffeomorphic to P_+. Since $A = \mathcal{G} \times_P (P/G_0)$, smooth sections of $\pi : A \to M$ are in bijective correspondence with P-equivariant smooth functions $\mathcal{G} \to P/G_0$, so these can be viewed as functions with values in P_+. The equivariancy condition reads as $f(u \cdot (g_0 g_1)) = g_1^{-1} g_0^{-1} \cdot f(u)$. But starting from $\tilde{g}_1 G_0$, we get $g_1^{-1} g_0^{-1} \tilde{g}_1 G_0 = g_1^{-1} g_0^{-1} \tilde{g}_1 g_0 G_0$, and $g_1^{-1} g_0^{-1} \tilde{g}_1 g_0 \in P_+$. This completes the proof of the first claim.

Given one function $f : \mathcal{G} \to P_+$, of course any other such function can be uniquely written as $\hat{f} = fh$ for a smooth function $h : \mathcal{G} \to P_+$, so it remains to understand P-equivariance. What we assume is that $f(u \cdot (g_0 g_1)) = g_1^{-1} g_0^{-1} f(u) g_0$ and we want $\hat{f}$ to satisfy the analogous equivariancy condition. But this exactly requires that $g_1^{-1} g_0^{-1} f(u) g_0 h(u \cdot (g_0 g_1)) = g_1^{-1} g_0^{-1} f(u) h(u) g_0$, which is equivalent to the claimed equivariancy of h. □

To connect to the well-known affine structure on the set of Weyl structures, we observe two alternative ways to express things using the exponential map. On the one hand, we have observed above that $\exp : \mathfrak{p}_+ \to P_+$ is a diffeomorphism. Thus we can write $h(u) = \exp(\Upsilon(u))$ and equivariancy of h is equivalent to $\Upsilon(u \cdot (g_0 g_1)) = \mathrm{Ad}(g_0)^{-1}(\Upsilon(u))$. On the other hand, in the proof of Theorem 3.1.3 of [16] it shown that also $(Z_1, \dots, Z_k) \mapsto \exp(Z_1) \cdots \exp(Z_k)$ defines a diffeomorphism $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \to P_+$. Correspondingly, we can write $h(u) = \exp(\Upsilon_1(u)) \cdots \exp(\Upsilon_k(u))$ where $\Upsilon_i : \mathcal{G} \to \mathfrak{g}_i$ is a smooth map for each $i = 1, \dots, k$. Again, equivariancy of h translates to $\Upsilon_i(u \cdot (g_0 g_1)) = \mathrm{Ad}(g_0)^{-1}(\Upsilon_i(u))$ for each i.

There is also a nice global way to express the affine structure. The filtration of TM induced by a parabolic geometry dualizes to a filtration of the cotangent bundle T^*M and we can form the associated graded bundle $\mathrm{gr}(T^*M)$. The general theory implies that this can be realized as $\mathcal{G} \times_P \mathrm{gr}(\mathfrak{p}_+) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{p}_+$.

Proposition 2.3. *Let $\pi : A \to M$ be the bundle of Weyl structures associated to a parabolic geometry $(p : \mathcal{G} \to M, \omega)$. Then for any smooth section s of π, there is an induced diffeomorphism $\varphi_s : T^*M \to A$.*

Proof. Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be the G_0-equivariant section determined by s. Since $\exp : \mathfrak{p}_+ \to P_+$ is a diffeomorphism, we conclude that $\Phi_s(u_0, Z) := \sigma(u_0)\exp(Z)$ defines a diffeomorphism $\Phi_s : \mathcal{G}_0 \times \mathfrak{p}_+ \to \mathcal{G}$. Given $g_0 \in G_0$, the definition readily implies that $\Phi_s(u \cdot g_0, \mathrm{Ad}(g_0^{-1})(Z)) = \Phi_s(u_0, Z) \cdot g_0$. Hence there is an induced diffeomorphisms between the orbit spaces $\mathrm{gr}(T^*M) = \mathcal{G}_0 \times_{G_0} \mathfrak{p}_+$ and $A = \mathcal{G}_0/G_0$. □

2.3. The basic geometric structures on A

It was shown in [21] that the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ gives rise to a connection on the tangent bundle TA of A. The argument used to obtain this connection is rather intricate: There is the opposite parabolic subgroup P^{op} to P which corresponds to the Lie subalgebra $\mathfrak{g}_- \oplus \mathfrak{g}_0 \subset \mathfrak{g}$ and one considers the homogeneous space G/P^{op}. Restricting the G-action to P and forming the associated bundle $\mathcal{G} \times_P (G/P^{op})$ the Cartan connection ω induces a natural affine connection on the total space of this bundle. It is then easy to see that $P \cap P^{op} = G_0$, so acting with P on eP^{op} defines a P-equivariant open embedding $A \to \mathcal{G} \times_P (G/P^{op})$, thus providing a connection on TA as claimed. Our first main result provides a more conceptual description of this connection, which directly implies compatibility with several additional geometric structures on A.

Proposition 2.4. *The canonical projection $\mathcal{G} \to A$ is a G_0-principal bundle and ω defines a Cartan connection on that bundle, so $(\mathcal{G} \to A, \omega)$ is a Cartan geometry of type (G, G_0). In particular, the $\mathfrak{g}_0$-component of ω defines a canonical principal connection on $\mathcal{G} \to A$ and $TA \cong \mathcal{G} \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0)$, so this inherits a canonical linear connection. Finally, there is a natural splitting $TA = L^- \oplus L^+$ into a direct sum of two subbundles of rank $\dim(M)$, which is parallel for the connection and such that L^+ is the vertical bundle of π.*

Proof. Mapping $u \in \mathcal{G}$ to the class of (u, eG_0) in $\mathcal{G} \times_P (P/G_0)$ is immediately seen to be surjective and its fibers coincide with the orbits of G_0 on $\mathcal{G}$. Hence one obtains an identification of $\mathcal{G}/G_0$ with A, and it is well known that this makes the projection $\mathcal{G} \to A$ into a G_0-principal bundle. The defining properties of ω for the group P and the Lie algebra $\mathfrak{p}$ then imply the corresponding properties for the group G_0 and the Lie algebra $\mathfrak{g}_0$, so ω defines a Cartan connection on $\mathcal{G} \to A$.

As a representation of G_0, we get $\mathfrak{g} = \mathfrak{g}_0 \oplus (\mathfrak{g}_- \oplus \mathfrak{p}_+)$. This means that we have given a G_0-invariant complement to $\mathfrak{g}_0$ in $\mathfrak{g}$. Decomposing ω accordingly, the component ω_0 in $\mathfrak{g}_0$ is G_0-equivariant, thus defining a principal connection on $\mathcal{G} \to A$, which induces linear connections on all associated vector bundles. Moreover, since ω is a Cartan connection on $\mathcal{G} \to A$, we can identify TA with the associated vector bundle

$$\mathcal{G} \times_{G_0} (\mathfrak{g}/\mathfrak{g}_0) \cong \mathcal{G} \times_{G_0} (\mathfrak{g}_- \oplus \mathfrak{p}_+).$$

This readily implies both the existence of a natural connection and of a compatible decomposition of TA with $L^- = \mathcal{G} \times_{G_0} \mathfrak{g}_-$ and $L^+ = \mathcal{G} \times_{G_0} \mathfrak{p}_+$. The tangent map $T\pi : TA \to TM$ is induced by the projection $\mathfrak{g}/\mathfrak{g}_0 \to \mathfrak{g}/\mathfrak{p}$.

Identifying $\mathfrak{g}/\mathfrak{g}_0$ with $\mathfrak{g}_- \oplus \mathfrak{p}_+$ the kernel of this projection is $\mathfrak{p}_+$, which shows that $L^+ \subset TA$ coincides with $\ker(T\pi)$. □

This result also gives us a basic supply of natural vector bundles on A, namely the vector bundles associated to the principal bundle $\mathcal{G} \to A$ via representations of G_0. Moreover, the principal connection on that bundle coming from ω_0 gives rise to an induced linear connection on each of these associated bundles. We will denote all these induced connections by D. Given an associated bundle $E \to A$, we can view D as an operator $D : \Gamma(E) \to \Gamma(T^*A \otimes E)$. Of course the splitting $TA = L^- \oplus L^+$ from Theorem 2.4 induces an analogous splitting of T^*A, which allows us to split D into two partial connections $D = D^- \oplus D^+$. Here $D^\pm : \Gamma(E) \to \Gamma((L^\pm)^* \otimes E)$. Viewing D as a covariant derivative, $D^\pm$ is defined by differentiating only in directions of the corresponding subbundle of TA.

2.4. Relations between natural vector bundles

Recall that the natural vector bundles for the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ are the associated vector bundles of the form $\mathcal{V}M = \mathcal{G} \times_P \mathbb{V}$ for representations $\mathbb{V}$ of P. Throughout this article, we will only consider the case that the center $Z(G_0)$ of the subgroup $G_0 \subset P$ acts diagonalizably on $\mathbb{V}$. Together with the fact that G_0 is reductive, this implies that $\mathbb{V}$ is completely reducible as a representation of G_0. One important subclass of natural bundles is formed by *completely reducible* bundles that correspond to representations of P on which the subgroup $P_+ \subset P$ acts trivially, which is equivalent to complete reducibility as a representation of P. On the other hand, there are *tractor bundles* which correspond to restrictions to P of representations of G.

Any representation $\mathbb{V}$ of P can naturally be endowed with a P-invariant filtration of the form $\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset \cdots \supset \mathbb{V}^N$ as follows (see Section 3.2.12 of [16]). The smallest component $\mathbb{V}^N$ consists of those elements, on which $\mathfrak{p}_+$ acts trivially under the infinitesimal action. The larger components are characterized iteratively by the fact that $v \in \mathbb{V}^j$ if and only if it is sent to $\mathbb{V}^{j+1}$ by the action of any element of $\mathfrak{p}_+$. Then one defines the *associated graded* representation $\mathrm{gr}(\mathbb{V}) := \oplus_{i=0}^N \mathrm{gr}_i(\mathbb{V})$ with $\mathrm{gr}_i(\mathbb{V}) := (\mathbb{V}^i/\mathbb{V}^{i+1})$ and $\mathbb{V}^{N+1} = \{0\}$. By construction, this is a completely reducible representation of P.

As an important special case, consider the restriction of the adjoint representation of G to P. Then it turns out that, up to a shift in degree, the canonical P-invariant filtration is exactly the filtration $\{\mathfrak{g}^i\}$ derived from the $|k|$-grading of $\mathfrak{g}$ as in Section 2.1. In particular, this implies that $\mathfrak{g}^2 = [\mathfrak{p}_+, \mathfrak{p}_+]$ and similarly, the higher filtrations components form the lower central series of $\mathfrak{p}_+$. Using this it is easy to see that the natural filtration on any representation $\mathbb{V}$ of P has the property that $\mathfrak{g}^i \cdot \mathbb{V}^j \subset \mathbb{V}^{i+j}$ for all $i, j \geq 0$ under the infinitesimal representation of $\mathfrak{p} = \mathfrak{g}^0$. This readily implies that there is a natural action of the associated graded $\mathrm{gr}(\mathfrak{p})$ on $\mathrm{gr}(\mathbb{V})$, which is compatible with the grading. Since the filtration of $\mathfrak{p}$ is induced by the

(non-negative part of the) grading on $\mathfrak{g}$, we can identify $\mathrm{gr}(\mathfrak{p})$ with $\mathfrak{p}$ via the inclusion of $\mathfrak{g}_i$ into $\mathfrak{g}^i$. Altogether, we get, for each $i, j \geq 0$, bilinear maps $\mathfrak{g}_i \times \mathrm{gr}_j(\mathbb{V}) \to \mathrm{gr}_{i+j}(\mathbb{V})$, which are P-equivariant (with trivial action of P_+) by construction.

As a representation of the subgroup $G_0 \subset P$, the associated graded $\mathrm{gr}(\mathbb{V})$ is isomorphic to $\mathbb{V}$. Indeed, we have observed above that $\mathbb{V}$ is completely reducible as a representation of G_0, so the same holds for each of the subrepresentations $\mathbb{V}^j \subset \mathbb{V}$. In particular, there always is a G_0-invariant complement $\mathbb{V}_j$ to the invariant subspace $\mathbb{V}^{j+1} \subset \mathbb{V}^j$ and we put $\mathbb{V}_N = \mathbb{V}^N$. By construction, we on the one hand get $\mathbb{V} \cong \oplus \mathbb{V}_j$ and on the other hand $\mathbb{V}_j \cong \mathbb{V}^j/\mathbb{V}^{j+1}$ which implies that claimed statement. Otherwise put, one can interpret the passage from $\mathbb{V}$ to $\mathrm{gr}(\mathbb{V})$ as keeping the restriction to G_0 of the P-action on $\mathbb{V}$ and extending this by the trivial action of P_+ to a new action of P.

The construction of the associated graded has a direct counterpart on the level of associated bundles. Putting $\mathcal{V}M := \mathcal{G} \times_P \mathbb{V} \to M$, any of the filtration components $\mathbb{V}^i$ defines a smooth subbundle $\mathcal{V}^i M := \mathcal{G} \times_P \mathbb{V}^i \to M$. Thus $\mathcal{V}M$ is filtered by the smooth subbundles $\mathcal{V}^i M$ and we can form the associated graded vector bundle $\mathrm{gr}(\mathcal{V}M) = \oplus(\mathcal{V}^i M/\mathcal{V}^{i+1}M)$. It is easy to see that this can be identified with the associated bundle $\mathcal{G} \times_P \mathrm{gr}(\mathbb{V})$. However, the fact that $\mathbb{V}$ and $\mathrm{gr}(\mathbb{V})$ are isomorphic as representations of G_0 does not have a geometric interpretation without making additional choices. Hence on the level of associated bundles, it is very important to carefully distinguish between a filtered vector bundle and its associated graded.

Any representation $\mathbb{V}$ of P defines a representation of G_0 by restriction. Hence denoting by $\pi : A \to M$ the bundle of Weyl structures, $\mathbb{V}$ also gives rise to a natural vector bundle over A that we denote by $\mathcal{V}A := \mathcal{G} \times_{G_0} \mathbb{V} \to A$. Some information is lost in that process, however, for example $\mathcal{G} \times_{G_0} \mathbb{V} \cong \mathcal{G} \times_{G_0} \mathrm{gr}(\mathbb{V})$ for any representation $\mathbb{V}$ of P. Next, sections of $\mathcal{V}A \to A$ can be naturally identified with smooth functions $\mathcal{G} \to \mathbb{V}$ that are G_0-equivariant. Similarly, sections of $\mathcal{V}M \to M$ are in bijective correspondence with smooth functions $\mathcal{G} \to \mathbb{V}$, which are P-equivariant. Thus we see that there is a natural inclusion of $\Gamma(\mathcal{V}M \to M)$ as a linear subspace of $\Gamma(\mathcal{V}A \to A)$. We will denote this by putting a tilde over the name of a section of $\mathcal{V}M \to M$ in order to indicate the corresponding section of $\mathcal{V}A \to A$. So both the sections of $\mathcal{V}M \to M$ and of its associated graded vector bundle can be interpreted as (different) subspaces of the space of sections of $\mathcal{V}A \to A$.

Now we can describe the relations of bundles and sections explicitly.

Theorem 2.5. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and let $\pi : A \to M$ be the corresponding bundle of Weyl structures. Fix a representation $\mathbb{V}$ of P and consider the corresponding natural bundles $\mathcal{V}M = \mathcal{G} \times_P \mathbb{V} \to M$ and $\mathcal{V}A = \mathcal{G} \times_{G_0} \mathbb{V} \to A$. Then we have:*

(1) $\mathcal{V}A$ can be naturally identified with the pullback bundle $\pi^*\mathcal{V}M$. In particular, $L^- \cong \pi^*TM$ and $L^+ \cong \pi^*T^*M$.

(2) The operation $\sigma \mapsto \tilde{\sigma}$ identifies $\Gamma(\mathcal{V}M \to M)$ with the subspace of $\Gamma(\mathcal{V}A \to A)$ consisting of those sections τ for which $D^+_\varphi \tau = -\varphi \bullet \tau$ for

all $\varphi \in \Gamma(L^+)$. Here $\bullet : L^+ \times \mathcal{V}A \to \mathcal{V}A$ is induced by the infinitesimal representation $\mathfrak{p}_+ \times \mathbb{V} \to \mathbb{V}$.

In particular, for a completely reducible bundle $\mathcal{V}M$, $\tilde\sigma = \pi^*\sigma$ and the image consists of all sections that are parallel for D^+.

*(3) Any section $s : M \to A$ of π determines a natural pullback operator $s^* : \Gamma(\mathcal{V}A \to A) \to \Gamma(\mathrm{gr}(\mathcal{V}M) \to M)$. In particular, choosing s, $\sigma \mapsto s^*\tilde\sigma$ defines a map $\Gamma(\mathcal{V}M) \to \Gamma(\mathrm{gr}(\mathcal{V}M))$. This map is induced by a vector bundle isomorphism $\mathcal{V}M \to \mathrm{gr}(\mathcal{V}M)$ that coincides with the isomorphism determined by the Weyl structure corresponding to s as in Section 5.1.3 of [**16**].*

Proof. (1) follows directly from the construction: Mapping a G_0-orbit in $\mathcal{G} \times \mathbb{V}$ to the P-orbit it generates, defines a bundle map $\mathcal{V}A \to \mathcal{V}M$ with base map $\pi : A \to M$. This evidently restricts to a linear isomorphism in each fiber and hence defines an isomorphism $\mathcal{V}A \to \pi^*\mathcal{V}M$. The second statement follows from the well known facts that $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ and $T^*M \cong \mathcal{G} \times_P \mathfrak{p}_+$ and the fact that $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_-$ as a representation of G_0.

(2) Since P is a semi-direct product, P-equivariancy of a function is equivalent to equivariancy under G_0 and P_+ and equivariancy under P_+ is equivalent to equivariancy for the infinitesimal action of $\mathfrak{p}_+$. Hence for a G_0-equivariant function $f : \mathcal{G} \to \mathbb{V}$, P-equivariancy is equivalent to the fact that for each $u \in \mathcal{G}$ and $Z \in \mathfrak{p}_+$ with fundamental vector field ζ_Z, we get $\zeta_Z(u)\cdot f = Z \cdot f(u)$. Here in the left hand side the vector field differentiates the function, while in the right hand side we use the infinitesimal representation of $\mathfrak{p}_+$ on $\mathbb{V}$. Suppose that u projects to $y \in A$. Then by definition, $\zeta_Z(u)$ is the horizontal lift with respect to D of a tangent vector $\varphi \in L^+_y \subset T_yA$. Hence $\zeta_Z(u) \cdot f$ represents $D_\varphi\tau(y) = D^+_\varphi\tau(y)$ in the, while $Z \cdot f(u)$ of course represents $\varphi \bullet \tau(y)$.

In the case of a completely reducible bundle, $\bullet$ is the zero map, so we see that our subspace coincides with the D^+-parallel sections. On the other hand, for any section $\sigma \in \Gamma(\mathcal{V}M)$, the pullback $\pi^*\sigma$ is constant along the fibers of π. Since we know from Theorem 2.4 that L^+ is the vertical subbundle of π, this implies that $\pi^*\sigma = \tilde\sigma$.

(3) As we have noted already, for any representation $\mathbb{V}$ of P, the associated graded $\mathrm{gr}(\mathbb{V})$ is isomorphic to $\mathbb{V}$ as a representation of G_0. Thus we conclude from (1) that we can not only identify $\mathcal{V}A$ with the pullback of $\mathcal{V}M$ but also with the pullback of the associated graded vector bundle $\mathrm{gr}(\mathcal{V}M)$. Hence for a smooth section $s : M \to A$ and a point $x \in M$, we can naturally identify the fiber $\mathcal{V}_{s(x)}A$ with the fiber over x of $\mathrm{gr}(\mathcal{V}M)$. This provides a pullback operator $s^* : \Gamma(\mathcal{V}A) \to \Gamma(\mathrm{gr}(\mathcal{V}M))$, so $\sigma \mapsto s^*\tilde\sigma$ defines an operator $\Gamma(\mathcal{V}M) \to \Gamma(\mathrm{gr}(\mathcal{V}M))$. This operator is evidently linear over $C^\infty(M, \mathbb{R})$ and thus induced by a vector bundle homomorphism $\mathcal{V}M \to \mathrm{gr}(\mathcal{V}M)$ with base map id_M. Suppose that for $\sigma \in \Gamma(\mathcal{V}M)$ and $x \in M$ we have $(s^*\tilde\sigma)(x) = 0$. Then the function $f : \mathcal{G} \to \mathbb{V}$ which corresponds to both σ and $\tilde\sigma$ has to vanish along the fiber of $\mathcal{G} \to A$ over $s(x)$. But P-equivariancy then implies that f vanishes along the fiber of $\mathcal{G} \to M$ over x, so $\sigma(x) = 0$. This implies that our bundle map is injective in each fiber and since both bundles have the same rank it is an isomorphism of vector bundles.

The standard description of the isomorphism $\mathcal{V}M \to \mathrm{gr}(\mathcal{V}M)$ induced by a Weyl structure is actually also phrased in the language of sections; Given the P-equivariant function $f : \mathcal{G} \to \mathbb{V}$ corresponding to σ and the G_0-equivariant section $\overline{s} : \mathcal{G}_0 \to \mathcal{G}$ determined by s, one considers the G_0-equivariant function $f \circ \overline{s}$. This describes a section of $\mathcal{G}_0 \times_{G_0} \mathbb{V} \cong \mathcal{G}_0 \times_{G_0} \mathrm{gr}(\mathbb{V})$. Going through the identifications, it is clear that this coincides with the isomorphism described above. □

Remark 2.6. (1) In principle, the pullback operation defined in part (3) of Theorem 2.5 could also be interpreted as having values in $\Gamma(\mathcal{V}M \to M)$. Since $\mathcal{V}A$ does not contain any information about the $\mathfrak{p}_+$-action on $\mathbb{V}$, the interpretation with values in $\Gamma(\mathrm{gr}(\mathcal{V}M) \to M)$ seems much more natural to us.

(2) The comparison to the standard description of Weyl structures in part (3) of the theorem also implies how the isomorphisms $\mathcal{V}M \to \mathrm{gr}(\mathcal{V}M)$ induced by sections s of $A \to M$ are compatible with the affine structure on the space of these sections from Theorem 2.2, compare with Proposition 5.1.5 of [16]. It is also easy to give a direct proof of this result in our picture. One just has to interpret the affine structure in terms of sections of $L^+ \to A$ and then use the obvious solution of the differential equation $D^+_\varphi \tau = \varphi \bullet \tau$ for appropriate sections φ.

2.5. The Weyl connections

We next describe the interpretation of Weyl connections in our picture. At the same time, we obtain a nice description of the Rho-corrected derivative associated to a Weyl structure, that was first introduced in [10], see Section 5.1.9 of [16] for a discussion. The Rho-corrected derivative comes from a principal connection on $\mathcal{G}$ determined by an equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$. One takes the component of ω in $\mathfrak{p}$ along the image of σ and extends it equivariantly to a principal connection. The name "Rho-corrected derivative" comes from the explicit formula of this derivative in terms of Weyl connection an the Rho-tensor. To obtain our description, we first observe that the pullback operation from part (3) of Theorem 2.5 clearly extends to differential forms with values in a natural vector bundle. Let $\mathbb{V}$ be a representation of P with corresponding natural bundles $\mathcal{V}M \to M$ and $\mathcal{V}A \to A$. Then one can pull back a $\mathcal{V}A$-valued k form φ on A along a section $s : M \to A$ to a $\mathrm{gr}(\mathcal{V}M)$-valued k-form $s^*\varphi$ on M in an obvious way.

Theorem 2.7. *Let $\mathbb{V}$ be a representation of P and let $\mathcal{V}M \to M$ and $\mathcal{V}A \to A$ be the corresponding natural bundles. For $\sigma \in \Gamma(\mathcal{V}M)$ consider the natural lift $\tilde{\sigma} \in \Gamma(\mathcal{V}A)$. For a smooth section $s : M \to A$ let ∇^s be the Weyl connection of the Weyl structure determined by s. Let $\xi \in \mathfrak{X}(M)$ be a vector field with natural lift $\tilde{\xi} \in \Gamma(L^-)$.*

(1) The pullback $s^*D\tilde{\sigma} \in \Omega^1(M, \mathrm{gr}(\mathcal{V}M))$ of $D\tilde{\sigma} \in \Omega^1(A, \mathcal{V}A)$ coincides with the image of $\nabla^s\sigma \in \Omega^1(M, \mathcal{V}M)$ under the isomorphism $\mathcal{V}M \to \mathrm{gr}(\mathcal{V}M)$ induced by s as in Theorem 2.5.

(2) The pullback $s^*(D^-_{\tilde\xi}\tilde\sigma) \in \Gamma(\mathrm{gr}(\mathcal{V}M))$ coincides with the image of the Rho-corrected derivative $\nabla^P_\xi\sigma \in \Gamma(\mathcal{V}M)$ under the isomorphism induced by s as in Theorem 2.5.

Proof. Let $\bar s : \mathcal{G}_0 \to \mathcal{G}$ be the G_0-equivariant section corresponding to s. For a point $x \in M$, an element $u \in \mathcal{G}$ with $p(u) = x$ lies in the image of $\bar s$ if and only if u projects to $s(x) \in A$. Assuming this, put $u_0 = q(u)$ where $q : \mathcal{G} \to \mathcal{G}_0$ is the projection, so $u = \bar s(u_0)$. To compute ∇^s, we need the horizontal lift $\hat\xi \in \mathfrak{X}(\mathcal{G}_0)$ of ξ for the principal connection $\bar s^*\omega_0$. This is characterized by the fact that $\hat\xi(u_0)$ projects onto $\xi(x)$ and that $\omega(u)(T_{u_0}\bar s \cdot \hat\xi(u_0))$ has vanishing $\mathfrak{g}_0$-component. But by construction $T_{u_0}\bar s \cdot \hat\xi(u_0)$ projects onto $T_x s \cdot \xi(x)$ and so vanishing of the $\mathfrak{g}_0$-component implies that this is the horizontal lift of $T_x s \cdot \xi(x)$ in u corresponding to the principal connection ω_0 that induces D. From this, (1) follows immediately.

The argument for (2) is closely similar. By definition, $\tilde\xi(s(x))$ is the unique tangent vector that lies in L^- and projects onto $\xi(x)$. The D-horizontal lift of this tangent vector in u, by construction, is mapped to $\mathfrak{g}_-$ by ω and projects onto $\xi(x) \in T_xM$. But this is exactly the characterizing property of the horizontal lift with respect to the principal connection $\gamma^{\bar s}$ used in Section 5.1.9 of [16] to define the Rho-corrected derivative. Thus the restriction of the G_0-equivariant function $\mathcal{G} \to \mathbb{V}$ representing $D^-_{\tilde\xi}\tilde\sigma$ to $\bar s(\mathcal{G}_0)$ coincides with the restriction of the P-equivariant function representing $\nabla^P_\xi\sigma$ and the claim follows from Theorem 2.5. □

2.6. The universal Rho-tensor

Using the pullback of bundle valued forms, we can also describe the Rho tensor in our picture. Recall that we use the convention of [15] and [16] for Rho tensors in the setting of general parabolic geometries, which differ by sign from the standard conventions for projective and conformal structures.

Proposition 2.8. *Let us view the projection $TA \to L^+$ as $P \in \Omega^1(A, L^+)$. Then for each smooth section $s : M \to A$, the pullback $s^*P \in \Omega^1(M, \mathrm{gr}(T^*M))$ coincides with the Rho-tensor of the Weyl structure determined by s as defined in Section 5.1.2 of [16].*

Proof. Take a point $x \in M$, a tangent vector $\xi \in T_xM$ and consider $T_x s \cdot \xi \in T_{s(x)}A$. Choose a point $u \in \mathcal{G}$ over $s(x)$ and consider its image $u_0 = q(u) \in \mathcal{G}_0$. Since u projects to $s(x)$, it lies in the image of the G_0-equivariant section $\bar s : \mathcal{G}_0 \to \mathcal{G}$ determined by s, so $u = \bar s(u_0)$. Taking a tangent vector $\hat\xi \in T_{u_0}\mathcal{G}_0$, the tangent vector $T_{u_0}\bar s \cdot \hat\xi \in T_u\mathcal{G}$, by construction, projects onto $T_x s \cdot \xi \in T_{s(x)}A$. But then, by definition, the L^+ component of $T_x s \cdot \xi$ is obtained by projecting $\omega(u)^{-1}(\omega_+(T_{u_0}\bar s \cdot \hat\xi))$ to $T_{s(x)}A$, where ω_+ denotes the $\mathfrak{p}_+$-component of the Cartan connection ω. But the Rho-tensor of $\bar s$ is defined as the $\mathrm{gr}(T^*M)$-valued form induced by the G_0-equivariant form $\bar s^*\omega_+$, which completes the argument. □

Definition 2.9. The form $\mathsf{P} \in \Omega^1(A, L^+)$ defined by the projection $TA \to L^+$ is called the *universal Rho-tensor* of the parabolic geometry $(p : \mathcal{G} \to M, \omega)$.

2.7. Curvature and torsion quantities

The curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the Cartan connection ω is defined by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$ for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. Since K is horizontal and P-equivariant, it can be interpreted as $\kappa \in \Omega^2(M, \mathcal{A}M)$, where $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ is the adjoint tractor bundle. In the same way, we can interpret it as a two-form on A with values in the associated bundle $\mathcal{G} \times_{G_0} \mathfrak{g}$. Since K is horizontal over M, it follows that this two form vanishes upon insertion of one tangent vector from $L^+ \subset TA$. In view of the G_0-invariant decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ we can decompose that two-form further. To do this, we denote by $\mathrm{End}_0(TA)$ the associated bundle $\mathcal{G} \times_{G_0} \mathfrak{g}_0$. Via the adjoint action, this can naturally be viewed as a subbundle of $L(TA, TA)$.

Definition 2.10. The components $T \in \Omega^2(A, L^-)$, $W \in \Omega^2(A, \mathrm{End}_0(TA))$ and $Y \in \Omega^2(A, L^+)$ of the two form on A induced by K are called the *universal torsion*, the *universal Weyl curvature* and the *universal Cotton-York tensor* of the parabolic geometry $(p : \mathcal{G} \to M, \omega)$.

The following result follows directly from the definitions.

Proposition 2.11. *For any smooth section $s : M \to A$, the pullbacks $s^*T \in \Omega^2(M, \mathrm{gr}(TM))$, $s^*W \in \Omega^2(M, \mathrm{End}_0(TM))$ and $s^*Y \in \Omega^2(M, \mathrm{gr}(T^*M))$ correspond to the components of the Cartan curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ under the isomorphism $\mathcal{A}M \cong \mathrm{gr}(\mathcal{A}M) \cong \mathrm{gr}(TM) \oplus \mathrm{End}_0(TM) \oplus \mathrm{gr}(T^*M)$ induced by the Weyl structure determined by s.*

These quantities are related to data associated to the Weyl structure determined by s in Section 5.2.9 of [16] and these results can be easily recovered in the current context.

On the level of A, the best way to interpret the components of the Cartan curvature is via the torsion and curvature of the canonical connection D. This interpretation will also be crucial for the analysis of the intrinsic geometric structure on A in Section 3 below. To formulate the result, we need a bit more notation. The Lie bracket is a G-equivariant, skew symmetric bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Now we can restrict this to entries from $\mathfrak{g}_- \oplus \mathfrak{p}_+$ and then decompose the values according to $\mathfrak{g} = (\mathfrak{g}_- \oplus \mathfrak{p}_+) \oplus \mathfrak{g}_0$, and the result will still be G_0-equivariant. The first component induces a two-form on A with values in TA which we denote by $\{\ ,\ \}$. Similarly, the $\mathfrak{g}_0$-component of the bracket defines a two-form $\{\ ,\ \}_0$ on A with values in $\mathrm{End}_0(TA)$. Using this, we formulate

Theorem 2.12. *Let $A \to M$ be the bundle of Weyl structures associated to a parabolic geometry, and let D be the canonical connection on TA. Let $\tau \in \Omega^2(A, TA)$ be the torsion and $\rho \in \Omega^2(A, L(TA, TA))$ be the curvature of D. Then we have:*

(1) The TA-valued two form $\tau + \{\ ,\ \}$ vanishes upon insertion of one section of L^+. On $\Lambda^2 L^-$, its components in L^- and L^+ are the tensors T and Y from Theorem 2.10, respectively.

(2) The curvature ρ has values in $\mathrm{End}_0(TA) \subset L(TA, TA)$. Moreover, $\rho + \{\ ,\ \}_0$ vanishes upon insertion of one section of L^+ and coincides with the tensor W from Theorem 2.10 on $\Lambda^2 L^-$.

Proof. This follows from well known results on the curvature and torsion of the affine connection induced by a reductive Cartan geometry. For $\xi \in \mathfrak{X}(A)$, let $\xi^h \in \mathfrak{X}(\mathcal{G})$ be the horizontal lift. Then $\omega(\xi^h) : \mathcal{G} \to (\mathfrak{g}_- \oplus \mathfrak{p}_+)$ is the equivariant function corresponding to ξ. Taking a second field η, the bracket $[\xi^h, \eta^h]$ lifts $[\xi, \eta]$ so $\omega_\pm([\xi^h, \eta^h])$ is the equivariant function representing $[\xi, \eta]$.

(1) From these considerations and the definition of the exterior derivative, it follows readily that $d\omega_\pm(\xi^h, \eta^h)$ is the equivariant function representing $\tau(\xi, \eta)$. On the other hand, the component of $[\omega(\xi^h), \omega(\eta^h)]$ in $\mathfrak{g}_- \oplus \mathfrak{p}_+$ of course represents $\{\xi, \eta\}$, so the claim follows from the definition of the curvature of a Cartan connection.

(2) It is also well known that $-\omega_0([\xi^h, \eta^h])$ is the function representing $\rho(\xi, \eta)$. Since the $\mathfrak{g}_0$-component of $[\omega(\xi^h), \omega(\eta^h)]$ clearly represents $\{\xi, \eta\}_0$, the result again follows from the definition of the Cartan curvature. □

3. The natural almost bi-Lagrangian structure

From here on, we take a different point of view. We study the geometry on the total space of the bundle of Weyl structures associated to a parabolic geometry from an intrinsic point of view, using the relation to parabolic geometries and Weyl structures as technical input. We shall see below that these structures become rather exotic in the case of general gradings, so we will restrict to parabolic geometries associated to $|1|$-gradings soon.

3.1. The almost bi-Lagrangian structure and torsion freeness

Consider a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of some type (G, P) and let $\pi : A \to M$ the associated bundle of Weyl structures. As we have noted in Section 2.2, the tangent bundle TA decomposes as $L^- \oplus L^+$, where $L^- = \mathcal{G} \times_{G_0} \mathfrak{g}_-$ and $L^+ = \mathcal{G} \times_{G_0} \mathfrak{p}_+$. It is also well known that $\mathfrak{g}_-$ and $\mathfrak{p}_+$ are dual as representations of G_0 via the restriction of the Killing form of $\mathfrak{g}$. Thus we obtain a non-degenerate pairing B mapping $L^- \times L^+$ to the trivial real line bundle $M \times \mathbb{R}$. This pairing can be extended as either a skew symmetric or a symmetric bilinear bundle map on TA, thus defining $\Omega \in \Omega^2(A)$ and $h \in \Gamma(S^2 T^* A)$. By construction, for each $y \in A$ both values $\Omega(y)$ and $h(y)$ are non-degenerate bilinear forms on $T_y A$ for which L^+_y and L^-_y are isotropic. The resulting structure (Ω, L^+, L^-) is called an *almost bi-Lagrangian structure.*

In particular, $\Omega \in \Omega^2(A)$ is an almost symplectic structure and an obvious first question is when this structure is symplectic, i.e. when $d\Omega = 0$.

Theorem 3.1. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and $\pi : A \to M$ its associated bundle of Weyl structures. Then the natural 2-form $\Omega \in \Omega^2(A)$ is closed if and only if (G, P) corresponds to a $|1|$-grading and the Cartan geometry $(p : \mathcal{G} \to M, \omega)$ is torsion-free.*

Proof. Let D be the canonical connection on TA from Section 2.3. Since Ω is induced by a G_0-invariant pairing on $\mathfrak{g}_- \oplus \mathfrak{g}_+$ it satisfies $D\Omega = 0$. If D were torsion-free, then $d\Omega$ would coincide with the complete alternation of $D\Omega$ and thus would vanish, too. In the presence of torsion, there still is a relation as follows. Expanding $D\Omega = 0$ by inserting vector fields $\xi, \eta, \zeta \in \mathfrak{X}(A)$, we obtain

$$0 = \xi \cdot \Omega(\eta, \zeta) - \Omega(D_\xi \eta, \zeta) - \Omega(\eta, D_\xi \zeta).$$

Now one takes the sum of the right hand side over all cyclic permutations of the arguments and uses skew symmetry of Ω to bring all derivatives of vector fields into the first component. Then one may use the definition of the torsion τ of D to rewrite $D_\xi \eta - D_\eta \xi$ as $[\xi, \eta] + \tau(\xi, \eta)$ and similarly for other combinations of the fields. Then the terms in which one field differentiates the value of Ω together with the terms involving a Lie bracket add up to the exterior derivative. One concludes that $D\Omega = 0$ implies

$$d\Omega(\xi, \eta, \zeta) = \textstyle\sum_{\mathrm{cycl}} \Omega(\tau(\xi, \eta), \zeta),$$

where in the right hand side we have the sum over all cyclic permutations of the arguments. Now let us assume that (G, P) corresponds to a $|k|$-grading with $k > 1$. Then L^- and L^+ decompose into direct sums of subbundles according to the grading of $\mathfrak{g}_-$ and $\mathfrak{p}_+$, respectively. Now we take $\xi \in L^-$ of degree -1, $\eta \in L^+$ of degree $i > 1$ and $\zeta \in L^+$ of degree $i - 1$. Then by Theorem 2.12, τ coincides with $\{\ ,\ \}$ on any two of these three fields. The restriction of $d\Omega$ to the subbundles corresponding to these three degrees is induced by the trilinear map $\mathfrak{g}_{-1} \times \mathfrak{g}_i \times \mathfrak{g}_{i-1} \to \mathbb{R}$ given by $(X, Y, Z) \mapsto \sum_{\mathrm{cycl}} B([X, Y], Z)$, where B denotes the Killing form of $\mathfrak{g}$. But $B([X, Y], Z)$ is already totally skew, so $d\Omega = 0$ would imply that $B([X, Y], Z) = 0$ for all elements of the given homogeneities. But non-degeneracy of B shows that $B([X, Y], Z) = 0$ for all Z implies $[X, Y] = 0$ while for $Y \in \mathfrak{g}_i$, the equation $[X, Y] = 0$ for all $X \in \mathfrak{g}_{-1}$ implies $Y = 0$, see Proposition 3.1.2 in [**16**].

Thus we may assume from now on that (G, P) corresponds to a $|1|$-grading. In this case, the bracket $\{\ ,\ \}$ is identically zero, so by Theorem 2.12, τ vanishes upon insertion of one element from L^+. Hence we see that $d\Omega$ vanishes upon insertion of two elements of L^+. Decomposing $\Lambda^3 T^*A$ according to $TA = L^- \oplus L^+$, the only potentially non-zero components of $d\Omega$ thus are the ones in $\Lambda^3(L^-)^*$ and in $\Lambda^2(L^-)^* \otimes (L^+)^*$.

Now if $\xi, \eta \in \Gamma(L^-)$ and $\zeta \in \Gamma(L^+)$, then we simply obtain $d\Omega(\xi, \eta, \zeta) = \Omega(T(\xi, \eta), \zeta)$, where T is defined in Theorem 2.10. Non-degeneracy of Ω shows that this vanishes for all ξ, η, ζ if and only if $T = 0$. This shows that vanishing of T is a necessary condition for Ω being closed. In the case of a $|1|$-grading, the pullback of T along a Weyl structure as in Theorem 2.11

is independent of the Weyl structure and gives the torsion of the Cartan geometry $(p : \mathcal{G} \to M, \omega)$.

To complete the proof, we thus have to show that (still in the case of a $|1|$-grading) vanishing of T implies that the component of $d\Omega$ in $\Lambda^3(L^-)^*$ vanishes identically. As above, Theorem 2.12 shows that this component is given by the sum of $\Omega(Y(\xi,\eta),\zeta)$ over all cyclic permutations of its arguments. But by construction, this is simply the complete alternation of Y, viewed as a section of $\Lambda^2 L^+ \otimes L^+$ via the identification $(L^-)^* \cong L^+$. In terms of the Cartan geometry $(p : \mathcal{G} \to M)$, it thus suffices to show that the component κ_+ of the Cartan curvature in $\mathfrak{p}_+$ always has trivial complete alternation.

To do this, we first observe that for a $|1|$-graded Lie algebra $\mathfrak{g}$, the subalgebra $\mathfrak{g}_0$ always splits into its center $\mathfrak{z}(\mathfrak{g}_0)$, which has dimension one, and a semisimple part $\mathfrak{g}_0^{ss}$. For a torsion-free geometry, the component κ_0 of κ with values in $\mathfrak{g}_0$ is the lowest non-vanishing homogeneous component of κ, which implies that its values have to lie in $\mathfrak{g}_0^{ss}$, compare with Theorem 4.1.1 in [16]. Thus we conclude that, viewed as a function $\mathcal{G} \to \mathfrak{g}$, κ has values in the subspace $\mathfrak{g}_0^{ss} \oplus \mathfrak{g}_1$.

Now we can apply the Bianchi identity in the form of equation (1.25) in Proposition 1.5.9 of [16]. This contains four terms, three of which are evaluations of the function κ or its derivative along some vector field, so these have values in $\mathfrak{g}_0^{ss} \oplus \mathfrak{g}_1$, too. Formulated in terms of functions, the Bianchi identity thus implies that for X_1, X_2, X_3 the cyclic sum over the arguments of $[X_1, \kappa(\omega^{-1}(X_2), \omega^{-1}(X_3))]$ has trivial component in $\mathfrak{z}(\mathfrak{g}_0)$. Now we can replace the X_i by their components in $\mathfrak{g}_-$ without changing the $\mathfrak{g}_0$-component of $[X_1, \kappa(\omega^{-1}(X_2), \omega^{-1}(X_3))]$, which in addition depends only on the $\mathfrak{g}_1$-component of κ. Now it is well known that for $X \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$, the component of $[X, Z]$ in $\mathfrak{z}(\mathfrak{g}_0)$ is a non-zero multiple of $B(X, Z)$, where B denotes the Killing form. But this exactly shows that, up to a non-zero factor, the $\mathfrak{z}(\mathfrak{g}_0)$-component of $\sum_{\text{cycl}}[X_1, \kappa(\omega^{-1}(X_2), \omega^{-1}(X_3))]$ represents the action of the complete alternation of κ_+ on the three vector fields corresponding to the X_i. Thus this complete alternation vanishes identically. □

Remark 3.2. (1) The failure of closedness of Ω for $|k|$-gradings with $k > 1$ can be described more precisely. The map $(X, Y, Z) \mapsto B([X, Y], Z)$ that shows up in the proof defines a G-invariant element in $\Lambda^3\mathfrak{g}^*$ and hence a bi-invariant 3-form on G. Restricting this to $\Lambda^3(\mathfrak{g}_- \oplus \mathfrak{p}_+)^*$, one obtains a G_0-invariant trilinear form, which is non-zero provided that $k > 1$. This in turn induces a natural 3-form on each manifold endowed with a Cartan Geometry of type (G, G_0). On a bundle of Weyl structures, the proof of Theorem 3.1 shows that this form always is a component of $d\Omega$.

(2) The parabolic geometries corresponding to $|1|$-gradings form a very interesting class of structures. For a $|1|$-grading, the subalgebras $\mathfrak{g}_-$ and $\mathfrak{p}_+$ become Abelian, whence the name "Abelian parabolic geometries" is sometimes used for these structures. The classification of $|1|$-gradings of simple Lie algebras is well known from the theory of Hermitian symmetric spaces, which motivates the more common name "AHS structures" where AHS is shorthand for "almost Hermitian symmetric".

Suppose that (G,P) corresponds to a $|1|$-grading on $\mathfrak{g}$. As noted in Section 2.1, the underlying structure $p_0 : \mathcal{G}_0 \to M$ of a Cartan geometry $(p : \mathcal{G} \to M)$ simply becomes a reduction of the linear frame bundle of M to the structure group $G_0 \subset GL(\mathfrak{g}_{-1})$. Thus AHS structures are a special class of G-structures, whose relevance is explained by the classification results by S. Kobayashi and T. Nagano in [24]. They prove that these are the only structures for which the group acts irreducibly, and which have the property that any automorphism is determined by a finite jet in a point but not by the one-jet in a point. In fact, automorphisms are always determined by the two-jet in a point and the equivalent canonical Cartan geometry of type (G,P) is the most effective description for these structures.

The torsion-freeness condition that shows up in Theorem 3.1 has a natural interpretation in the language of G_0-structures. As noted in the proof, the torsion T associated to a Weyl structure in this case is independent of the Weyl structure. It turns out that this coincides with the intrinsic torsion of the G_0-structure (i.e. the component of the torsion that is independent of the choice of connection). Thus torsion-freeness of the Cartan geometry corresponds to the usual notion of integrability in the language of G_0-structures.

(3) For some types of AHS-structures, torsion-freeness implies local flatness. Locally flat structures can be equivalently be characterized as being obtained from local charts with values in the homogeneous model G/P, for which the transition functions are given by restrictions of left actions of elements of g. This case anyway plays a very important role in the results we are going to prove, so our results are also relevant to these types of AHS structures.

3.2. Local frames

From this point on, we restrict the discussion to torsion-free geometries of some type (G,P) that corresponds to a $|1|$-grading of $\mathfrak{g}$, so that by Theorem 3.1 Ω defines a symplectic structure on A. Recall from Section 2.4 that any vector field $\xi \in \mathfrak{X}(M)$ determines a section $\tilde{\xi} \in \Gamma(L^-)$, and since $\mathfrak{g}_-$ is a completely reducible representation in the $|1|$-graded case, we get $D^+\tilde{\xi} = 0$. Similarly, a one-form $\alpha \in \Omega^1(M)$ defines a section $\tilde{\alpha} \in \Gamma(L^+)$ such that $D^+\tilde{\alpha} = 0$. We further know that $L^- \cong \pi^*TM$ and $L^+ = \pi^*T^*M$. This implies that starting with local frames for TM and T^*M defined on some open set $U \subset M$, the lifts form local frames for $L^\pm$ defined on $\pi^{-1}(U)$, so together, these form a local frame for TA. One may in particular use dual local frames for TM and T^*M in which case the resulting local frame for TA is nicely adapted to the almost bi-Lagrangian structure and thus both to Ω and to h. As a preparation for the following computations, we next compute the Lie brackets of such sections.

Proposition 3.3. *Consider a torsion-free AHS structure $(p : \mathcal{G} \to M, \omega)$ and let $\pi : A \to M$ be the corresponding bundle of Weyl structures. Let $\xi, \eta \in \mathfrak{X}(M)$ be vector fields and $\alpha, \beta \in \Omega^1(M)$ be one-forms on M and consider the corresponding sections $\tilde{\xi}, \tilde{\eta} \in \Gamma(L^-)$ and $\tilde{\alpha}, \tilde{\beta} \in \Gamma(L^+)$.*

Then for the Lie brackets on A, we get $[\tilde\alpha,\tilde\beta]=0$ and $[\tilde\xi,\tilde\alpha]=D_{\tilde\xi}\tilde\alpha\in\Gamma(L^+)$. Finally, the L^--component of $[\tilde\xi,\tilde\eta]$ coincides with $\widetilde{[\xi,\eta]}$, while its L^+-component coincides with $-Y(\tilde\xi,\tilde\eta)$, see Theorem 2.10.

Proof. By definition of the torsion

$$\tau(X,Z)=D_XZ-D_ZX-[X,Z] \tag{1}$$

for all $X,Z\in\mathfrak{X}(A)$. If at least one of the two fields is a section of L^+, then the left hand side of (1) vanishes by Theorem 2.12. Moreover, all the sections coming from M are parallel in L^+-directions. This immediately shows that $[\tilde\alpha,\tilde\beta]=0$ and $0=D_{\tilde\xi}\tilde\alpha-[\tilde\xi,\tilde\alpha]$. In view of torsion-freeness, Theorem 2.12 further tells us that $\tau(\tilde\xi,\tilde\eta)=Y(\tilde\xi,\tilde\eta)\in\Gamma(L^+)$. Inserting $X=\tilde\xi$ and $Z=\tilde\eta$ into the right hand side of (1), the first two terms are sections of L^-, so the claim about the L^+-component of $[\tilde\xi,\tilde\eta]$ follows. Finally, since $\tilde\xi$ and $\tilde\eta$ are lifts to A of ξ and η, the bracket $[\tilde\xi,\tilde\eta]\in\mathfrak{X}(A)$ is a lift of $[\xi,\eta]$. Since $\widetilde{[\xi,\eta]}$ is the unique section of L^- that projects onto $[\xi,\eta]$, it has to coincide with the L^--component of that lift. □

In particular, we see that, while L^+ always defines an involutive distribution, L^- is only involutive if the curvature component Y from Theorem 2.10 vanishes identically. From the interpretation via the Cartan curvature, one easily concludes that this is equivalent to local vanishing of the Cartan curvature. Thus our structure is bi-Lagrangian (in the sense that both subbundles $L^\pm$ are integrable) if and only if the initial parabolic geometry is locally flat.

3.3. The canonical metric

We next study the pseudo-Riemannian metric h induced on A. By definition, the subbundles $L^\pm$ are isotropic for h, so this metric always has split signature (n,n), where $n=\dim(M)$. Our main next aim will be to prove that the metric h is always Einstein. As a first step in this direction, consider the canonical connection D and its curvature $\rho\in\Omega^2(A,\mathrm{End}_0(TA))$ as described in Section 2.7.

Lemma 3.4. *The Ricci-type contraction of ρ is a non-zero multiple of h.*

Proof. By Theorem 2.12, $\rho+\{\ ,\ \}_0$ vanishes upon insertion of one section of L^+ and coincides with W on $\Lambda^2(L^-)^*$. Decomposing Λ^2TA^* according to $TA=L^+\oplus L^-$, we conclude that the component of ρ in $\Lambda^2(L^+)^*$ vanishes, its component in $(L^-)^*\otimes(L^+)^*$ is induced by $-\{\ ,\ \}_0$, and the component in $\Lambda^2(L^-)^*$ is induced by W. On the other hand, $\mathrm{End}_0(TA)=\mathcal{G}\times_{G_0}\mathfrak{g}_0$, so this is a subbundle of $((L^-)^*\otimes L^-)\oplus((L^+)^*\otimes L^+)\subset TA^*\otimes TA$. Thus we conclude that the Ricci-type contraction of ρ vanishes on $L^+\times L^+$, while its components on $L^-\times L^+$ and $L^-\times L^-$ are induced by the Ricci-type contractions of $-\{\ ,\ \}_0$ and W, respectively. By Theorem 2.11, the pullback of W along any section $s:M\to A$ represents the Weyl curvature of the Weyl structure determined by s. In the torsion-free case, this is well known to have values in an irreducible representation of G_0 that occurs with multiplicity one in $\Lambda^*\mathfrak{g}_-^*\otimes\mathfrak{g}$, which implies that any contraction of W vanishes identically.

Hence we see that the Ricci type contraction of ρ has values in $(L^-)^* \otimes (L^+)^*$ and is induced by the Ricci type contraction of $-\{\ ,\ \}_0$, so this is a natural bundle map, and we can compute it on the inducing representations. Take a basis $\{e_i\}$ of $\mathfrak{g}_{-1}$ and let $\{e^i\}$ be the dual basis of $\mathfrak{g}_{-1}^* \cong \mathfrak{g}_1$, which means that for the Killing form B, we get $B(e_i, e^j) = \delta_i^j$. Now we have to view the $\mathfrak{g}_0$ component of the bracket $[\ ,\ ]$ in $\mathfrak{g}$ as a map sending $(\mathfrak{g}_- \oplus \mathfrak{p}_+)^2$ to an endomorphism of $\mathfrak{g}_- \oplus \mathfrak{p}_+$ via the adjoint action. Hence for $X, Y \in \mathfrak{g}_{-1}$ and $Z, W \in \mathfrak{g}_1$, the Ricci type contraction sends $\binom{X}{Z}$ and $\binom{Y}{W}$ to

$$\textstyle\sum_i B([[\binom{X}{Z}, \binom{e_i}{0}], \binom{Y}{W}], \binom{0}{e^i}) + \sum_i B([[\binom{X}{Z}, \binom{0}{e^i}], \binom{Y}{W}], \binom{e_i}{0}).$$

Expanding the first sum using invariance of the Killing form and the fact that $\mathfrak{g}_{-1}$ is abelian, we obtain

$$\textstyle\sum_i B([[Z, e_i], Y], e^i) = \sum_i B(Z, [e_i, [Y, e^i]]) = \sum_i B(Z, [Y, [e_i, e^i]]),$$

and in the same way the second sum gives $\sum_i B([X, [e_i, e^i]], W)$. But the element $\sum_i [e_i, e^i] \in \mathfrak{g}_0$ is obtained from the identity map in $\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}^*$ via the isomorphism to $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1$ and the bracket in $\mathfrak{g}$. Since these both are $\mathfrak{g}_0$-equivariant, $\sum_i [e_i, e^i]$ is $\mathfrak{g}_0$-invariant and thus contained in the center of $\mathfrak{g}_0$. In the $|1|$-graded case, this center is spanned by the grading element E. In addition, $B(E, \sum_i [e_i, e^i]) = \sum_i B([E, e_i], e^i) = -\dim(\mathfrak{g}_{-1})$, so $\sum_i [e_i, e^i]$ is a non-zero multiple of E. Hence the whole contraction gives a non-zero multiple of $B(Z, Y) + B(X, W) = h(\binom{X}{Z}, \binom{Y}{W})$. □

Now by construction, the canonical connection D satisfies $Dh = 0$, so D is metric for h. This implies that the Levi-Civita connection ∇ of h can be computed from D and its torsion τ. Indeed, we claim that for $\xi, \eta, \zeta \in \mathfrak{X}(A)$, $h(\nabla_\xi \eta, \zeta)$ is given by

$$h(D_\xi \eta, \zeta) - \tfrac{1}{2} h(\tau(\xi, \eta), \zeta) + \tfrac{1}{2} h(\tau(\xi, \zeta), \eta) + \tfrac{1}{2} h(\tau(\eta, \zeta), \xi). \tag{2}$$

This evidently defines a linear connection ∇ on TA. Moreover, the last three terms in (2) are visibly skew symmetric in η and ζ, whence the fact that D is metric with respect to h implies that ∇ is metric with respect to h, too. On the other hand, since the last two terms in (2) are symmetric in ξ and η, and τ is the torsion of D, one immediately verifies that ∇ is torsion-free. Let us write $C \in \Gamma(\otimes^2 T^*A \otimes TA)$ for the contorsion tensor between ∇ and D, so $C(\xi, \eta) = \nabla_\xi \eta - D_\xi \eta$ and the last three terms in (2) explicitly express $h(C(\xi, \eta), \zeta)$. Using this, we prove the following result

Theorem 3.5. *For any torsion-free AHS structure, the pseudo-Riemannian metric h induced by the canonical almost bi-Lagrangian structure on the bundle A of Weyl structures is an Einstein metric with non-zero scalar curvature.*

Proof. Theorem 2.12 in the torsion-free case shows that τ vanishes upon insertion of one section of L^+ and has values in L^+. Thus equation (2) shows that $h(C(\xi, \eta), \zeta)$ vanishes if one of the three fields is a section of L^+. This shows that the only non-zero component of C is the one mapping $L^- \times L^-$ to L^+. Now it is standard how to compute the curvature of ∇ from C and the curvature ρ of D via differentiating the equation defining C. The result

contains terms in which C is differentiated as well as terms in which values of C are inserted into C. From the form of C we have just deduced, it follows that the latter terms vanish identically.

Using this, one computes that for $\xi, \eta, \zeta \in \mathfrak{X}(A)$ the difference $R(\xi,\eta)(\zeta) - \rho(\xi,\eta)(\zeta)$ is given by

$$D_\xi(C(\eta,\zeta)) - D_\eta(C(\xi,\zeta)) + C(\xi, D_\eta\zeta) - C(\eta, D_\xi\zeta) - C([\xi,\eta],\zeta). \tag{3}$$

(This is just the covariant exterior derivative of C with respect to D evaluated on ξ and η and then applied to ζ.) In view of Theorem 3.4, it suffices to prove that the Ricci-type contraction of this expression vanishes. To compute this contraction, we leave ξ and ζ as entries, insert the elements of a local frame of TA for η and hook the result into h together with the elements of the dual frame. First of all, (3) visibly vanishes for $\zeta \in \Gamma(L^+)$. If we insert for η an element of a frame for L^-, then the element of the dual frame will sit in L^+. Since C has values in L^+, these summands do not contribute to the contraction. Thus we only have to take into account the case that we insert elements of a frame for L^+ for η, and then the first and fourth term of (3) visibly vanish. The remaining three terms vanish if ξ is a section of L^+, so what we have to compute is

$$\textstyle\sum_i h\left(-D^+_{e^i}(C(\xi,\zeta)) + C(\xi, D^+_{e^i}\zeta) - C([\xi,e^i],\zeta), e_i\right)$$

for a smooth local frame $\{e^i\}$ for L^+ with dual frame $\{e_i\}$ for L^- and local sections $\xi, \zeta \in \Gamma(L^-)$. Now we can take ξ and ζ and the local frames to be obtained from vector fields respectively one-forms on M. Then $D^+_{e^i}\zeta = 0$, while $[\xi, e^i] \in \Gamma(L^+)$ by Theorem 3.3 and thus $C([\xi,e^i],\zeta) = 0$.

Thus we are left with computing $\sum_i h(D^+_{e^i}(C(\xi,\zeta)), e_i)$ with the frames, ξ and ζ all coming from M. In particular, e_i is parallel for D^+ so since D is metric for h, we may rewrite this as $\sum_i e^i \cdot h(C(\xi,\zeta), e_i)$. We can then insert the formula for $h(C(\xi,\zeta),e_i)$ resulting from (2), taking into account that on entries from L^- the torsion τ is determined by the tensor Y from Theorem 2.10. Viewing Y as a section of $\Lambda^2(L^-)^* \otimes (L^-)^*$, this leads to

$$\textstyle\sum_i e^i \cdot h(C(\xi,\zeta),e_i) = \frac{1}{2}\sum_i e^i \cdot (-Y(\xi,\zeta,e_i) + Y(\xi,e_i,\zeta) + Y(\zeta,e_i,\xi)).$$

From the proof of Theorem 3.1, we know that the complete alternation of Y vanishes, which allows us to rewrite this as $\sum_i e^i \cdot Y(\zeta,e_i,\xi)$. Under the standing assumption that all sections come from M, they are parallel for D^+, so we can complete the proof by showing that $\sum_i (D^+_{e^i}Y)(\zeta,e_i,\xi) = 0$.

Now by definition, Y is a component of the Cartan curvature, which descends to a well defined section of the bundle $\Lambda^2 T^*M \otimes \mathcal{A}M$, where $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$. By torsion freeness, the full Cartan curvature has the form $(0, W, Y)$ with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Hence by Theorem 2.5, we get

$$D^+_\varphi(0,W,Y) = (0, D^+_\varphi W, D^+_\varphi, Y) = -\varphi \bullet (0,W,Y),$$

and the action $\bullet$ is induced by the Lie bracket on $\mathfrak{g}$. Since this bracket vanishes on $\mathfrak{g}_1 \times \mathfrak{g}_1$ and defines the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{\pm 1}$, we conclude that $D^+W = 0$ and $D^+_\varphi Y = W(\varphi)$, where we view W as an section of $\Lambda^2(L^--)^* \otimes \mathrm{End}(L^+) \cong$

$\Lambda^2(L^--)^* \otimes (L^-)^* \otimes (L^+)^*$ in the right hand side. But this implies that $\sum_i (D^+_{e^i}Y)(\zeta, e_i, \xi)$ is given by evaluating a trace of W on ζ and ξ and thus vanishes. □

Remark 3.6. In the special case of a projective structure on a surface Σ, the resulting Einstein metric on the four-manifold A is also anti-self-dual, see [19] and also [8].

Remark 3.7. The automorphisms of a projective structure on a smooth manifold M lift to become isometric symplectomorphisms of (A, Ω, h), see [19]. For background about automorphisms of a parabolic geometry, see [1, 16, 20].

Remark 3.8. A pair (h, Ω), consisting of a split-signature metric h and a symplectic form Ω that are related by an endomorphism which squares to become the identity map, is also known as an *almost para-Kähler structure*. Here, following [6], we refer to such a pair, or rather its associated triple (Ω, L_+, L_-), as an almost bi-Lagrangian structure.

3.4. Geometry of Weyl structures

Viewed as a section of $\pi : A \to M$, any Weyl structure defines an embedding of M into A, and we can now study this embedding via submanifold geometry related to the almost bi-Lagrangian structure. In particular, we can pull back the two-form Ω and the pseudo-Riemannian metric to M along s, and this naturally leads to the following definitions.

Definition 3.9. Let $(p : \mathcal{G} \to M, \omega)$ be a torsion-free AHS structure and let $s : M \to A$ be a smooth section.

(1) The Weyl structure corresponding to s is called *Lagrangian* if and only if $s^*\Omega = 0$ and thus $s(M) \subset A$ is a Lagrangian submanifold.

(2) The Weyl structure corresponding to s is called *non-degenerate* if and only if $s^*h \in \Gamma(S^2T^*M)$ is non-degenerate and thus defines a pseudo-Riemannian metric on M.

These properties can easily be characterized in terms of the Rho tensor.

Proposition 3.10. *A Weyl structure is Lagrangian if and only if its Rho tensor is symmetric and non-degenerate if and only if the symmetric part of its Rho tensor is non-degenerate.*

Proof. For a point $x \in M$ and a tangent vector $\xi \in T_xM$ consider $T_xs \cdot \xi \in T_{s(x)}(A)$. Since this is a lift of ξ, its L^--component has to coincide with $\tilde{\xi}(s(x))$. On the other hand, by Theorem 2.8, the L^+-component of $T_xs \cdot \xi$ corresponds to $\mathsf{P}(x)(\xi) \in T^*_xM$. Pulling back the pairing between L^- and L^+, one thus obtains the map $(\xi, \eta) \mapsto \mathsf{P}(x)(\eta)(\xi)$ and thus the result follows from the definitions of Ω and h. □

Remark 3.11. (1) For any type of parabolic geometry, there are natural line bundles called *bundles of scales*, an example in the AHS-case is provided by the bundle $\mathcal{E}M$ in Theorem 4.2 below. If $\mathcal{E}M$ is any bundle of scales on M, then mapping a Weyl structure to the induced Weyl connection on

$\mathcal{E}M$ induces a bijective correspondence between Weyl structures and linear connections on $\mathcal{E}M$. Fixing $\mathcal{E}M$, one calls a Weyl structure *closed* if the corresponding linear connection on $\mathcal{E}M$ is flat and *exact* if in addition there is a global parallel section of $\mathcal{E}M$, see [15] and Section 5.1 of [16]. For general types of geometries there is a larger freedom of choice of bundles of scales, but for AHS-structures all bundles of scales lead to the same subclasses of closed and exact Weyl structures.

Together with the general theory of Weyl structures, Theorem 3.10 implies that, on a torsion-free AHS-structure, a Weyl structure is Lagrangian if and only if it is closed. This follows from the relation between curvature and torsion of a Weyl connection and the Cartan curvature as discussed in Example 5.2.3 of [16] in the setting of AHS structures. By Theorem 5.2.3 of that reference, the curvature $R \in \Omega^2(M, \mathcal{G} \times_g \mathfrak{g}_0)$ of the Weyl connection corresponding to $s \in \Gamma(A)$ can be computed as $s^*W + \partial s^*\mathsf{P}$ for the quantities from Theorem 2.11 and a certain natural bundle map ∂. For the action on a bundle of scales, the component of R with values in the center $\mathfrak{z}(\mathfrak{g}_0)$ is relevant. For a torsion free geometry, W is the lowest non-zero component of the Cartan curvature and hence by general results has values in an irreducible subrepresentation which cannot meet this center. Hence the curvature of the Weyl connection is only induced by $\partial s^*\mathsf{P}$, and the component of this in $\mathfrak{z}(\mathfrak{g}_0)$ is immediately seen to be the skew part of the Rho tensor up to a non-zero factor.

This result nicely corresponds to the fact that for the canonical symplectic structure on any cotangent bundle T^*N, the image of a one-form $\alpha \in \Omega^1(N)$ in T^*N is a Lagrangian submanifold if and only if $d\alpha = 0$.

(2) In the case of an AHS-structure, the cotangent bundle T^*M coincides with the associated graded bundle, so Theorem 2.3 shows that a Weyl structure s determines a diffeomorphism $\varphi_s : T^*M \to A$. Now we can use this to pull back the geometric structures on A to T^*M and in particular, in the torsion-free case, compare the pullback of the symplectic form $\Omega \in \Omega^2(A)$ to the canonical symplectic structure on T^*M. Recall that the diffeomorphism φ_s is induced by $\Phi_s : \mathcal{G}_0 \times \mathfrak{p}_+ \to \mathcal{G}$, $\Phi_s(u_0, Z) = \sigma(u_0) \cdot \exp(Z)$, where $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is the equivariant section determined by s.

Equivariancy of the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ then implies that the pullback $\Phi_s^*\omega$ can be easily expressed explicitly in terms of $\sigma^*\omega$. Denoting by $q : \mathcal{G}_0 \times \mathfrak{p}_+ \to T^*M$ the canonical projection, the definition of Ω in Section 3.1 shows that $q^*\varphi_s^*\Omega = \Phi_s^*\Omega$ sends tangent vectors ξ, η to the alternation of the pairing between $(\Phi_s^*\omega)_-(\xi) \in \mathfrak{g}_{-1}$ and $(\Phi_s^*\omega)_+(\eta) \in \mathfrak{g}_1$. On the other hand, it is easy to explicitly describe $q^*\alpha \in \Omega^1(\mathcal{G}_0 \times \mathfrak{p}_+)$, where $\alpha \in \Omega^1(T^*M)$ is the canonical one-form. From this, one can explicitly compute the pullback $-q^*d\alpha$ of the canonical symplectic form on T^*M and show that it equals the sum of $\Phi_s^*\Omega$ and the pullback of the alternation of the Rho-tensor. In particular, generalizing a result from [28] in the projective case, we conclude that $\varphi_s : T^*M \to A$ is a symplectomorphism if and only if the Weyl-structure s is Lagrangian. Indeed, it turns out that also the split-signature metric φ_s^*h

on T^*M can be computed explicitly in terms of the underlying AHS-structure. All this will be taken up in more detail elsewhere.

To start the geometric study of Lagrangian Weyl structures, we can characterize when s has the property that the submanifold $s(M) \subset A$ is totally geodesic.

Theorem 3.12. *Let $(p : \mathcal{G} \to M, \omega)$ be a torsion-free AHS structure and $\pi : A \to M$ its bundle of Weyl structures. Let $s : M \to A$ be a smooth section corresponding to a Lagrangian Weyl structure, let ∇^s denote the corresponding Weyl connections and P^s the corresponding Rho-tensor. Then the following conditions are equivalent:*

(1) The submanifold $s(M) \subset A$ is totally geodesic for the canonical connection D.

(2) The submanifold $s(M) \subset A$ is totally geodesic for the Levi-Civita connection of h.

(3) $\nabla^s \mathsf{P}^s = 0$

Proof. We will use abstract index notation to carry out the computations and denote the Rho tensor of s just by P, so this has the form P_{ij} and is symmetric by assumption. Since for each $y \in A$ and $x := \pi(y) \in M$, we can identify L^-_y with T_xM and L^+_y with T^*_xM, we can use the index notation also on A, but here tangent vectors have the form (ξ^i, α_j). In this language the proof of Theorem 3.10 shows that for $x \in M$ the tangent space $T_{s(x)}s(M)$ consists of all pairs of the form $(\xi^i, \mathsf{P}_{ja}\xi^a)$. The condition that $s(M)$ is totally geodesic with respect to D means that for vector field (ξ, α) on A that is tangent to $s(M)$ along $s(M)$, also the covariant derivative in directions tangent to $s(M)$ is tangent to $s(M)$.

In particular, for a vector field $\eta \in \mathfrak{X}(M)$, we know from above that $(\widetilde{\eta^j}, \widetilde{\mathsf{P}_{kb}\eta^b}) \in \mathfrak{X}(A)$ is tangent to $s(M)$ along $s(M)$. Since these fields are parallel for D^+, we see that $s(M)$ is totally geodesic for D if and only if all derivatives with respect to D of that field are tangent to $s(M)$ along $s(M)$. But this can be checked by pulling back the components of $D(\widetilde{\eta^j}, \widetilde{\mathsf{P}_{kb}\eta^b})$ along s, which by Theorem 2.7 leads to $\nabla^s_i\eta^j$ and

$$\nabla^s_i \mathsf{P}_{ka}\eta^a = \eta^a \nabla^s_i \mathsf{P}_{ka} + \mathsf{P}_{ka}\nabla^s_i\eta^a, \tag{4}$$

respectively. So evidently, the result is tangent to $s(M)$ if and only if $\eta^a\nabla^s_i\mathsf{P}_{ka} = 0$ and since this has to hold for each η, we conclude that (1) is equivalent to (3).

To deal with (2), we use the information on the contorsion tensor C from Section 3.3. As observed in the proof of Theorem 3.5, the only non-zero component of C maps $L^- \times L^-$ to L^+. This means that $(\widetilde{\eta^j}, \widetilde{\mathsf{P}_{kb}\eta^b})$ is also parallel in L^+-directions for the Levi-Civita connection, so as above, we can use the pull back of the full derivative along s and the result has to be tangent to $s(M)$. We write the pullback of C along s as C_{ijk} using the convention that $C(\xi, \eta)_k = \xi^i\eta^j C_{ijk}$. Now formula (2) from Section 3.3 expresses C in terms of the torsion τ of D (with h just playing the role of identifying

L^+ with the dual of L^-) and we know that the torsion corresponds to the Cartan curvature quantity Y, see Theorem 2.12. Writing the pullback of this along s in abstract index notion as Y_{ijk}, we conclude form formula (2) that $C_{ijk} = \frac{1}{2}(-Y_{ijk} + Y_{ikj} + Y_{jki})$. The pullback of the derivative along s again has first component $\nabla^s_i \eta^j$ but for the second component, we have to add $C_{iak}\eta^a$ to the right hand side of (4).

But it is a well known fact (see Theorem 5.2.3 of [16]) that the pullback of Y along s is given by the covariant exterior derivative of the Rho-tensor of s and since ∇^s is torsion-free, this is expressed in abstract index notation as $Y_{ijk} = \nabla^s_i \mathsf{P}_{jk} - \nabla^s_j \mathsf{P}_{ik}$. Inserting this into the formula for C_{ijk}, we immediately conclude that we have to add $\eta^a \nabla^s_a \mathsf{P}_{ik} - \eta^a \nabla^s_k \mathsf{P}_{ia}$ to (4). As above, this implies that (2) is equivalent to

$$\nabla^s_i \mathsf{P}_{ka} + \nabla^s_a \mathsf{P}_{ik} - \nabla^s_k \mathsf{P}_{ia} = 0. \tag{5}$$

Of course, (5) is satisfied if $\nabla^s \mathsf{P} = 0$. Conversely, if (5) holds, then summing over all cyclic permutations of the indices shows that the total symmetrization of $\nabla^s \mathsf{P}$ has to vanish. But subtracting three times this total symmetrization from the left hand side of (5), one obtains $-2\nabla^s_k \mathsf{P}_{ia}$, so this has to vanish, too. □

The result of Theorem 3.12 is particularly interesting if s is non-degenerate. By Theorem 3.10 this implies that P^s defines a pseudo-Riemannian metric on M and since ∇^s is torsion-free, $\nabla^s \mathsf{P}^s = 0$ implies that ∇^s is the Levi-Civita connection of P^s. On the other hand, P^s is always related to the Ricci-type contraction of the curvature of ∇^s, see Section 4.1.1 of [16]. In particular, for projective structures, symmetry of P^s implies that it is a non-zero multiple of the Ricci curvature of ∇^s, see [2], so in this case P^s defines an Einstein metric on M. The condition that a projective structure contains the Levi-Civita connection of an Einstein metric can be expressed as a reduction of projective holonomy, see [12] and [11].

For a non-degenerate Lagrangian Weyl structure s, there is a well defined second fundamental form of $s(M)$ with respect to any linear connection on TA which is metric for h. Extending the result of Theorem 3.12 in this case, we can next explicitly compute the second fundamental forms for D and for the Levi-Civita connection. To formulate the result, we use abstract index notation as in the proof of Theorem 3.12.

Fix the section $s : M \to A$ corresponding to a non-degenerate, Lagrangian Weyl structure. By non-degeneracy, the Rho tensor P_{ij} of s admits an inverse $\mathsf{P}^{ij} \in \Gamma(S^2TM)$ which is characterized by $\mathsf{P}^{ij}\mathsf{P}_{jk} = \delta^i_k$. In the proof of Theorem 3.12, we have seen that $T_{s(x)}s(M)$ is spanned by all elements of the form $(\eta^i, \mathsf{P}_{ka}\eta^a)$ with $\eta^i \in T_xM$. The definition of h readily implies that the normal space $T^\perp_{s(x)}s(M)$ consists of all pairs of the form $(\eta^i, -\mathsf{P}_{jk}\eta^k)$, so we can identify both the tangent and the normal space in $s(x)$ with T_xM via projection to the first component. Correspondingly, the second fundamental form of $s(M)$ (with respect to any connection on TA which is metric for h) can be viewed as a $\binom{1}{2}$-tensor field on M. We denote the second fundamental

form of $s : M \to (A, h)$ with respect to D by II^s_D and with respect to the Levi-Civita connection of h by II^s_h.

Theorem 3.13. *Let $(p : \mathcal{G} \to M, \omega)$ be a torsion-free AHS-structure with bundle of Weyl structures $\pi : A \to M$. Let $s : M \to A$ be a non-degenerate, Lagrangian Weyl structure with Weyl connection ∇^s and Rho-tensor $P \in \Gamma(S^2T^*M)$, then*

$$\mathrm{II}^s_D = -\tfrac{1}{2}P^{ka}\nabla^s_i P_{ja} \qquad \text{and} \qquad \mathrm{II}^s_h = -\tfrac{1}{2}P^{ka}(\nabla^s_i P_{ja} + \nabla^s_j P_{ia} - \nabla^s_a P_{ij}).$$

Proof. We only have to project the derivatives computed in the proof of Theorem 3.12 to the normal space. From the description of tangent and normal spaces, it follows readily that projecting (ξ^i, α_j) to the normal space and taking the L^--component of the result, one obtains $\frac{1}{2}(\xi^i - \mathsf{P}^{ia}\alpha_a)$. Using this, the formulae follow directly from the the proof of Theorem 3.12. □

4. Relations to non-linear invariant PDE

We conclude this article by discussing a relation between the geometry on the bundle A of Weyl structures and non-linear invariant PDE associated to AHS structures. We will mainly consider the prototypical example of a projectively invariant PDE of Monge-Ampère type. We briefly discuss analogs of this and other invariant non-linear PDE for specific types of AHS structures, but this will be taken up in detail elsewhere.

4.1. A tractorial description of A

We start by deriving an alternative description of the bundle $A \to M$ of Weyl structures associated to an AHS structure $(p : \mathcal{G} \to M, \omega)$ based on tractor bundles. As mentioned in Section 2.4, these are bundles associated to representations of P that are restrictions of representations of G. An important feature of these bundles is that they inherit canonical linear connections from the Cartan connection ω. Together with some algebraic ingredients, these form the basis for the machinery of BGG sequences that was developed in [17] and [9], which will provide input to some of the further developments. We have also met the canonical invariant filtration on representations of P in Section 2.4 and the corresponding filtration of associated bundles by smooth subbundles. For representations of G, these admit a simpler description, that we derive first. In most of the examples we need below, this description is rather obvious, so readers not interested in representation theory aspects can safely skip the proof of this result.

Recall that for any $|k|$-grading on $\mathfrak{g}$, there is a unique grading element E, such that for $i = -k, \dots, k$ the subspace $\mathfrak{g}_i$ is the eigenspace with eigenvalue i for the adjoint action of E. In particular, E has to lie in the center of the subalgebra $\mathfrak{g}_0$. In the case of a $|1|$-grading, this center has dimension 1 and thus is spanned by E.

Lemma 4.1. *Consider a Lie group G with simple Lie algebra $\mathfrak{g}$ that is endowed with a $|1|$-grading with grading element E, and let $G_0 \subset P \subset G$ be subgroups associated to this grading. Let $\mathbb{V}$ be a representation of G*

which is irreducible as a representation of $\mathfrak{g}$. Then there is a G_0-invariant decomposition $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_N$ such that

- *Each $\mathbb{V}_j$ is an eigenspace of E.*
- *For $i \in \{-1, 0, 1\}$ and each j, we have $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$.*
- *For each $j > 0$, restriction of the representation defines a surjection $\mathfrak{g}_1 \otimes \mathbb{V}_{j-1} \to \mathbb{V}_j$.*
- *The canonical P-invariant filtration on $\mathbb{V}$ is given by $\mathbb{V}^j = \oplus_{\ell \geq j} \mathbb{V}_\ell$.*

Proof. Note first that the $|1|$-grading of $\mathfrak{g}$ induces a $|1|$-grading on the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$, which has the same grading element E as $\mathfrak{g}$. It is also well known that there is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ that contains E. Complexifying $\mathbb{V}$ and $\mathfrak{g}$ if necessary and then passing back to the E-invariant subspace $\mathbb{V}$, we conclude that E acts diagonalizably on $\mathbb{V}$. Denoting the λ-eigenspace for E in $\mathbb{V}$ by $\mathbb{V}_\lambda$, it follows readily that $\mathfrak{g}_i \cdot \mathbb{V}_\lambda \subset \mathbb{V}_{\lambda+i}$ for $i \in \{-1, 0, 1\}$. Now take an eigenvalue λ_0 with minimal real part, let N be the smallest positive integer such that $\lambda_0 + N + 1$ is not an eigenvalue of E and put $\mathbb{V}_j := \mathbb{V}_{\lambda_0 + j}$ for $j = 0, \dots, N$. Then, by construction, $\mathfrak{g}_{-1}$ acts trivially on $\mathbb{V}_0$, $\mathfrak{g}_1$ acts trivially on $\mathbb{V}_N$, and each $\mathbb{V}_j$ is $\mathfrak{g}_0$-invariant. This shows that $\mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_N$ is $\mathfrak{g}$-invariant and hence has to coincide with $\mathbb{V}$ by irreducibility.

By definition, the adjoint action of each element $g_0 \in G_0$ preserves the grading of $\mathfrak{g}$, which easily implies that $\mathrm{Ad}(g_0)(E)$ acts on $\mathfrak{g}_i$ by multiplication by i for $i \in \{-1, 0, 1\}$. This means that $\mathrm{Ad}(g_0)(E) - E$ lies in the center of $\mathfrak{g}$, so $\mathrm{Ad}(g_0)(E) = E$. But then for $v \in \mathbb{V}$, we can compute $E \cdot g_0 \cdot v$ as $\mathrm{Ad}(g_0)(E) \cdot g_0 \cdot v = g_0 \cdot E \cdot v$. This shows that each $\mathbb{V}_j$ is G_0-invariant and it only remains to prove the last two claimed properties of the decomposition.

We put $\tilde{\mathbb{V}}_0 := \mathbb{V}_0$ and for $j > 0$, we inductively define $\tilde{\mathbb{V}}_j$ as the image of the map $\mathfrak{g}_1 \otimes \tilde{\mathbb{V}}_{j-1} \to \mathbb{V}_j$. Then, by construction, each $\tilde{\mathbb{V}}_j$ is a $\mathfrak{g}_0$-invariant subspace of $\mathbb{V}_j$, so $\tilde{\mathbb{V}} := \oplus_{j=0}^N \tilde{\mathbb{V}}_j \subset \mathbb{V}$ is invariant under the actions of $\mathfrak{g}_0$ and $\mathfrak{g}_1$. But for $X \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$, we have $[X, Z] \in \mathfrak{g}_0$ and for $v \in \mathbb{V}$ we get

$$X \cdot Z \cdot v = Z \cdot X \cdot v + [Z, X] \cdot v.$$

This inductively shows that $\tilde{\mathbb{V}}$ is invariant under the action of $\mathfrak{g}_{-1}$. Thus it is $\mathfrak{g}$-invariant and hence has to coincide with $\mathbb{V}$ by irreducibility, so it remains to verify the claimed description of the canonical P-invariant filtration.

To do this, we first claim that an element $v \in \mathbb{V}$ such that $Z \cdot v = 0$ for all $Z \in \mathfrak{g}_1$ has to be contained in $\mathbb{V}_N$. It suffices to prove this for the complexification, so we may assume that both $\mathfrak{g}$ and $\mathbb{V}$ are complex and so there is a highest weight vector $v_0 \in \mathbb{V}$ which is unique up to scale by irreducibility. It is then well known that $\mathbb{V}$ is spanned by vectors obtained from v_0 by the iterated action of elements in negative root spaces of $\mathfrak{g}$. Since on such elements E has non-positive eigenvalues, we conclude that $v_0 \in \mathbb{V}_N$. Now assume that for some $j < N$, the space $W := \{v \in \mathbb{V}_j : \mathfrak{g}_1 \cdot v = \{0\}\}$ is non-trivial. Then, by construction, this is a $\mathfrak{g}_0$-invariant subspace of $\mathbb{V}_j$ on which the center of $\mathfrak{g}_0$ acts by a scalar, so it must contain a vector that is annihilated by all elements in positive root spaces of $\mathfrak{g}_0$. But since any positive root space of $\mathfrak{g}$ either is a positive root space of $\mathfrak{g}_0$ or is contained in

$\mathfrak{g}_1$, this has to be a highest weight vector for $\mathfrak{g}$, which contradicts uniqueness of v_0 up to scale.

Having proved the claim, we can first interpret it as showing that $\mathbb{V}^N = \mathbb{V}_N$. From this the description of the P-invariant filtration follows by backwards induction: Suppose that that we have shown that $\mathbb{V}^{N-j} = \mathbb{V}_{N-j} \oplus \cdots \oplus \mathbb{V}_N$ and let $w \in \mathbb{V}$ be such that for all $Z \in \mathfrak{g}_1$, we have $Z \cdot w \in \mathbb{V}^{N-j}$. Decomposing $w = w_0 + \cdots + w_N$, we conclude that for all $i < N - j - 1$ we must have $Z \cdot w_i = 0$ and hence $w \in \mathbb{V}_{N-j-1} \oplus \cdots \oplus \mathbb{V}_N$. Together with the obvious fact that $\oplus_{\ell \geq N-j-1} \mathbb{V}_\ell \subset \mathbb{V}^{N-j-1}$, this implies the description of the P-invariant filtration. □

Using this, we can now prove an alternative description of the bundle of Weyl structures that, as we shall see in the examples below, generalizes the construction of [19].

Theorem 4.2. *Suppose that (G, P) corresponds to a $|1|$-grading of the Lie algebra $\mathfrak{g}$ of G. Let $\mathbb{V}$ be a representation of G, which is non-trivial and irreducible as a representation of $\mathfrak{g}$, with natural P-invariant filtration $\{\mathbb{V}^j : j = 0, \dots, N\}$ such that $\mathbb{V}/\mathbb{V}^1$ has real dimension 1. For a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) let $\mathcal{V}M$ be the tractor bundle induced by $\mathbb{V}$, $\mathcal{V}^j M$ the subbundle corresponding to $\mathbb{V}^j$, and define $\mathcal{E}M$ to be the real line bundle $\mathcal{V}M/\mathcal{V}^1 M$.*

Then the bundle $A \to M$ of Weyl structures can be naturally identified with the open subbundle in the projectivization $\mathcal{P}(\mathcal{V}M/\mathcal{V}^2 M)$ formed by all lines that are transversal to the subbundle $\mathcal{V}^1 M/\mathcal{V}^2 M$ of hyperplanes. This in turn leads to an identification of $A \to M$ with the bundle of all linear connections on the line bundle $\mathcal{E}M \to M$.

Proof. By assumption $\mathbb{V}^1 \subset \mathbb{V}$ is a P-invariant hyperplane, so this descends to a P-invariant hyperplane $\mathbb{V}^1/\mathbb{V}^2$ in $\mathbb{V}/\mathbb{V}^2$. Passing to the projectivization, the complement of this hyperplane is a P-invariant open subset $U \subset \mathcal{P}(\mathbb{V}/\mathbb{V}^2)$ and hence defines a natural open subbundle in the associated bundle $\mathcal{P}(\mathcal{V}M/\mathcal{V}^2 M)$.

Now take the decomposition $\mathbb{V} = \oplus_{j=0}^N \mathbb{V}_j$ from Theorem 4.1. Then $\mathbb{V}_0$ is a line in $\mathbb{V}$ transversal to $\mathbb{V}^1$ and hence defines a point $\ell_0 \in U$. We claim that the P-orbit of ℓ_0 is all of U, while its stabilizer subgroup in P coincides with G_0. This shows that $U \cong P/G_0$ and thus implies the first claimed description of $A \to M$. As observed in Section 2.4, an element $g \in P$ can be written uniquely as $\exp(Z)g_0$ for $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$. We know that $\mathbb{V}_0$ is G_0-invariant, so for $w \in \ell_0$ we get $g_0 \cdot w = aw$ for some nonzero element $a \in \mathbb{R}$. On the other hand, $\exp(Z) \cdot w = w + Z \cdot w + \frac{1}{2} Z \cdot Z \cdot w + \dots$, and all but the first two summands lie in $\mathbb{V}^2$. This shows that the action of $\exp(Z)g_0$ sends ℓ_0 to the line in $\mathbb{V}/\mathbb{V}^2$ spanned by $w + Z \cdot w + \mathbb{V}^2$. But from Theorem 4.1, we know that the action defines a surjection $\mathfrak{g}_1 \otimes \mathbb{V}_0 \to \mathbb{V}_1$, which shows that $P \cdot \ell_0 = U$. On the other hand, it is well known that $\mathfrak{g}_1$ is an irreducible representation of $\mathfrak{g}_0$. Thus also $\mathfrak{g}_1 \otimes \mathbb{V}_0$ is irreducible, so $Z \cdot w = 0$ if and only $Z = 0$, which shows that the stabilizer of ℓ_0 in P coincides with G_0.

For the second description, we need some input from the machinery of BGG sequences. There is a natural invariant differential operator $S : \Gamma(\mathcal{E}M) \to \Gamma(\mathcal{V}M)$ which splits the tensorial map $\Gamma(\mathcal{V}M) \to \Gamma(\mathcal{E}M)$ induced by the quotient projection $\mathcal{V}M \to \mathcal{E}M$. It turns out that the operator $\Gamma(\mathcal{E}M) \to \Gamma(\mathcal{V}M/\mathcal{V}^{i+1}M)$ induced by S has order i, so there is an induced vector bundle map from the jet prolongation $J^i\mathcal{E}M$ to $\mathcal{V}M/\mathcal{V}^{i+1}M$. As proved in [5], the representation $\mathbb{V}$ determines an integer i_0 such that this is an isomorphism for all $i \leq i_0$. For a non-trivial representation $i_0 \geq 1$, so we conclude that $J^1\mathcal{E}M \cong \mathcal{V}M/\mathcal{V}^2M$. Since S splits the tensorial projection, we see that, in a point $x \in M$, the hyperplane $\mathcal{V}^1_xM/\mathcal{V}^2_xM$ corresponds to the jets of sections vanishing in x. Thus lines in $\mathcal{V}M/\mathcal{V}^2M$ that are transversal to $\mathcal{V}^1M/\mathcal{V}^2M$ exactly correspond to lines in $J^1\mathcal{E}M$ that are transversal to the kernel of the natural projection to $\mathcal{E}M$. Choosing such a line is equivalent to choosing a splitting of this projection and thus of the jet exact sequence for $J^1\mathcal{E}M$. It is well known that the choice of such a splitting is equivalent to the choice of a linear connection on $\mathcal{E}M$. □

Example 4.3. (1) **Oriented projective structures**. For $n \geq 2$, put $G := SL(n+1, \mathbb{R})$ and let P be the stabilizer of the ray in $\mathbb{R}^{n+1}$ spanned by the first element e_0 in the standard basis. Taking the complementary hyperplane spanned by the remaining basis vectors, one obtains a $|1|$-grading on the Lie algebra $\mathfrak{g}$ of G by decomposing into blocks of sizes 1 and n as in $\left(\begin{smallmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{smallmatrix}\right)$. The subgroup $G_0 \subset P$ is then easily seen to consist of all block diagonal matrices in P.

Now we define $\mathbb{V} := \mathbb{R}^{(n+1)*}$, the dual of the standard representation of G. In terms of the dual of the standard basis, this decomposes as the sum of $\mathbb{V}_0 := \mathbb{R} \cdot e_0^*$ and $\mathbb{V}_1$ spanned by the remaining basis vectors. All properties claimed in Theorem 4.1 are obviously satisfied in this case. The tractor bundle $\mathcal{V}M$ corresponding to $\mathbb{V}$ is usually called the (standard) cotractor bundle $\mathcal{T}^*M$ and the line bundle $\mathcal{E}M$ is the bundle $\mathcal{E}(1)$ of projective 1-densities. Since $\mathbb{V}^2 = \{0\}$ in this case, Theorem 4.2 realizes A as an open subbundle in $\mathcal{P}(\mathcal{T}^*M)$, and this is exactly the construction from [19]. In fact, it is well known that $\mathcal{T}^*M \cong J^1\mathcal{E}(1)$ in this case, this is even used as a definition in [2].

More generally, for $k \geq 2$, we can take $\mathbb{V}$ to be the symmetric power $S^k\mathbb{R}^{(n+1)*}$. This visibly decomposes as $\mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k$, where $\mathbb{V}_i$ is spanned by the symmetric products of $(e_0^*)^{k-i}$ with i other basis elements. This corresponds to the tractor bundle $S^k\mathcal{T}^*M$, while the quotient $\mathbb{V}/\mathbb{V}^1$ induces the kth power of $\mathcal{E}(1)$, which is usually denoted by $\mathcal{E}(k)$ and called the bundle of projective k-densities. Again, all properties claimed in Theorem 4.1 are obviously satisfied.

(2) **Conformal structures**. For $p+q = n \geq 3$, we put $G := SO(p+1, q+1)$ and we take a basis $e_0, \dots, e_{n+1}$ for the standard representation $\mathbb{R}^{n+2}$ of G such that the non-trivial inner products are $\langle e_0, e_{n+1}\rangle = 1$, and $\langle e_i, e_i\rangle = 1$ for $1 \leq i \leq p$ and $\langle e_i, e_i\rangle = -1$ for $p+1 \leq i \leq n$. Splitting matrices into blocks of sizes 1, n, and 1 defines a $|1|$-grading of $\mathfrak{g}$ according to $\left(\begin{smallmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & * \\ 0 & * & * \end{smallmatrix}\right)$,

where entries marked by $*$ are determined by other entries of the matrix. Again G_0 turns out to consist of block diagonal matrices and is isomorphic to the conformal group $CO(p+1,q+1)$ via the adjoint action on $\mathfrak{g}_{-1}$. In view of Theorem 3.2 we conclude that a parabolic geometry of type (G,P) on a manifold M is equivalent to a conformal structure.

The standard representation $\mathbb{V}$ of G now decomposes as $\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2$, with the subspaces spanned by e_0, $\{e_1,\dots,e_n\}$, and e_{n+1}, respectively. This induces the *standard tractor bundle* $\mathcal{T}M$ on conformal manifolds, and the bundle induced by $\mathbb{V}/\mathbb{V}^1$ is the bundle $\mathcal{E}[1]$ of *conformal* 1*-densities*. Hence Theorem 4.2 in this case realizes A as an open subbundle of the projectivization of the quotient of $\mathcal{T}M$ by its smallest filtration component. Again, it is well known that this quotient is isomorphic to $J^1\mathcal{E}[1]$.

Alternatively, for $k \geq 2$, we can take $\mathbb{V}$ to be the trace-free part $S^k_0\mathbb{R}^{n+2}$ in the symmetric power of the standard representation. This leads to the bundle $S^k_0\mathcal{T}M$ and the line bundle $\mathcal{E}[k]$ of conformal densities of weight k. Here the decomposition from Theorem 4.1 becomes a bit more complicated, since the individual pieces $\mathbb{V}_j$ are not irreducible representations of G_0 in general. Still the properties claimed in Theorem 4.1 are obvious via the construction from the decomposition of the standard representation.

(3) **Almost Grassmannian structures**. Here we choose integers $2 \leq p \leq q$, put $n = p+q$, take $G := SL(n,\mathbb{R})$ and $P \subset G$ the stabilizer of the subspace spanned by the first p vectors of the standard basis of the standard representation $\mathbb{R}^n$ of G. Fixing the complementary subspace spanned by the remaining q vectors in that basis, one obtains a decomposition of the Lie algebra $\mathfrak{g}$ of G into blocks of sizes p and q, which defines a $|1|$-grading as in the projective case. Then G_0 again turns out to consist of block diagonal matrices and hence is isomorphic to $S(GL(p,\mathbb{R}) \times GL(q,\mathbb{R}))$, while $\mathfrak{g}_{-1}$ can be identified with the space of $q \times p$-matrices endowed with the action of G_0 defined by matrix multiplication from both sides.

Hence the corresponding geometries exist in dimension pq and they are essentially given by an identification of the tangent bundle with a tensor product of two auxiliary bundles of rank p and q, respectively, see Section 4.1.3 of [**16**]. There it is also shown that for these types of structures the Cartan curvature has two fundamental components, but their nature depends on p and q. For $p = q = 2$, such a structure is equivalent to a split-signature conformal structure, so we will not discuss this case here. If $p = 2$ and $q > 2$, then one of these quantities is the intrinsic torsion of the structure, but the second is a curvature, so this is a case in which there are non-flat, torsion-free examples. For $p > 3$, the intrinsic torsion splits into two components, and torsion-freeness of a geometry implies local flatness.

The basic choice of a representation $\mathbb{V}$ that Theorem 4.2 can be applied to is given by $\Lambda^p\mathbb{R}^{n*}$, the pth exterior power of the dual of the standard representation. This decomposes as $\mathbb{V} = \mathbb{V}_0 \oplus \dots \oplus \mathbb{V}_p$, where $\mathbb{V}_j$ is spanned by wedge products of elements of the dual of the standard basis that contain $p-j$ factors from $\{e^*_1,\dots,e^*_p\}$. The properties claimed in Theorem 4.1

can be easily deduced from the construction from the dual of the standard representation.

4.2. The projective Monge-Ampère equation

This is the prototypical example of the non-linear PDE that we want to study. In the setting of projective geometry, we have met the density bundles $\mathcal{E}(k)$ for $k > 0$ in Theorem 4.3. We define $\mathcal{E}(-k)$ to be the line bundle dual to $\mathcal{E}(k)$ and use the convention that adding "(k)" to the name of a bundle indicates a tensor product with $\mathcal{E}(k)$ for $k \in \mathbb{Z}$. The first step towards the construction of the projective Monge-Ampère equation is that there is a projectively invariant, linear, second order differential operator $H : \Gamma(\mathcal{E}(1)) \to \Gamma(S^2T^*M(1))$ called the *projective Hessian*. Indeed, this is the first operator in the BGG sequence determined by the standard cotractor bundle, see [**13**].

Now for a section $\sigma \in \Gamma(\mathcal{E}(1))$, $H(\sigma)$ defines a symmetric bilinear form on each tangent space of M, and such a form has a well defined determinant. In projective geometry, this determinant admits an interpretation as a density as follows. In the setting of part (1) of Theorem 4.3, the top exterior power $\Lambda^{n+1}\mathbb{R}^{(n+1)*}$ is a trivial representation, which implies that the bundle $\Lambda^{n+1}\mathcal{J}^*M$ is canonically trivial. Identifying $\mathcal{J}^*M$ with $J^1\mathcal{E}(1)$, the jet exact sequence $0 \to T^*M(1) \to J^1\mathcal{E}(1) \to \mathcal{E}(1) \to 0$ implies that $\Lambda^{n+1}\mathcal{J}^*M \cong \Lambda^nT^*M(n+1)$, so $\Lambda^nT^*M \cong \mathcal{E}(-n-1)$. This isomorphism can be encoded as a tautological section of $\Lambda^nTM(-n-1)$. To form the determinant of $H(\sigma)$, one now takes the tensor product of two copies of this canonical section and of n copies of $H(\sigma)$ and forms the unique (potentially) non-trivial complete contraction of the result (so the two indices of each copy of $H(\sigma)$ have to be contracted into different copies of the tautological form). This shows that $\det(H(\sigma))$ can be naturally interpreted as a section of $\mathcal{E}(-n-2)$.

Assuming that $\sigma \in \Gamma(\mathcal{E}(1))$ is nowhere vanishing, we can form $\sigma^k \in \Gamma(\mathcal{E}(k))$ for any $k \in \mathbb{Z}$, and hence

$$\det(H(\sigma)) = \pm\sigma^{-n-2} \tag{6}$$

is a projectively invariant, fully non-linear PDE on nowhere vanishing sections of $\mathcal{E}(1)$. Observe that multiplying σ by a constant, the two sides of the equation scale by different powers of the constant, so allowing a constant factor instead of just a sign in the right hand side of the equation would only be a trivial modification.

4.3. Interpretation in terms of Weyl structures

Let us first observe that a nowhere vanishing section $\sigma \in \mathcal{E}(1)$ uniquely determines a Weyl structure. In the language of Theorem 4.2 this can be either described as the structure corresponding to the flat connection on $\mathcal{E}(1)$ determined by σ or as the one corresponding to the line in $\mathcal{J}^*M$ spanned by $S(\sigma)$, where S denotes the BGG splitting operator. From either interpretation it is clear that this Weyl structure remains unchanged if σ

is multiplied by a non-zero constant. Alternatively, one can easily verify that any projective class on M contains a unique connection such that σ is parallel for the induced connection on $\mathcal{E}(1)$.

To deal with non-flat cases in the following theorem, we use a concept of mean curvature tailored to the case of connections compatible with an almost bi-Lagrangian structure that was introduced in [18]. That article uses the terminology of (almost) para-Kähler structures, which is slightly different from ours, but it is easy to translate between the two.

Theorem 4.4. *Let M be an oriented smooth manifold of dimension n which is endowed with a projective structure. Let $\sigma \in \Gamma(\mathcal{E}(1))$ be a nowhere-vanishing section and let us denote by ∇^σ the Weyl connections of the Weyl structure determined by σ and by P^σ its Rho tensor. Then we have:*

(1) An appropriate constant multiple of σ satisfies (6) if and only if $\nabla^\sigma(\det(P^\sigma)) = 0$ and $\det(P^\sigma)$ is nowhere vanishing.

(2) The Weyl structure determined by σ is always Lagrangian. If M is projectively flat, then an appropriate constant multiple of σ satisfies (6) if and only if this Weyl structure is non-degenerate and the image of the corresponding section $s : M \to A$ is a minimal submanifold. This extends to curved projective structures provided that minimality of $s(M)$ is defined as vanishing of the mean curvature form associated to the canonical connection D via the definition in [18, p. 120].

Proof. It is well known how to decompose the curvature of a linear connection on TM into the projective Weyl curvature and the Rho-tensor, see Section 3.1 of [2] (taking into account the sign conventions mentioned in Section 2.6). It is also shown there that the Bianchi identity shows that the skew part of the Rho tensor is a non-zero multiple of the trace of the curvature tensor, which describes the action of the curvature on the top exterior power of the tangent bundle and thus on density bundles. This readily implies that P^σ is symmetric, so the Weyl structure defined by σ is Lagrangian by Theorem 3.10.

(1) It is well known that the projective Hessian in terms of a linear connection ∇ in the projective class and its Rho tensor P is given by the symmetrization of $\nabla^2\sigma - \mathsf{P}\sigma$, see Section 3.2 of [13]. (The different sign is caused by different sign conventions for the Rho-tensor.) But by definition $\nabla^\sigma\sigma = 0$ and P^σ is symmetric, so we conclude that $H(\sigma) = -\mathsf{P}^\sigma\sigma$ and hence $\det(H(\sigma)) = (-1)^n\sigma^n \det(\mathsf{P}^\sigma)$. Now for each $k \neq 0$, the sections of $\mathcal{E}(k)$ that are parallel for ∇^σ are exactly the constant multiples of σ^k, so (1) follows readily.

(2) Since P^σ is symmetric, $\det(\mathsf{P}^\sigma)$ is nowhere vanishing if and only if P^σ is non-degenerate. By Theorem 3.10, this is equivalent to non-degeneracy of the Weyl structure determined by σ, and we assume this from now on. Let us write $\det(\mathsf{P}^\sigma)$ in terms of the tautological section ϵ of $\Lambda^n TM(-n-1)$ as above as $C(\epsilon \otimes \epsilon \otimes (\mathsf{P}^\sigma)^{\otimes^n})$, where C denotes the appropriate contraction. Applying ∇^σ to this, we observe that C and ϵ are projectively invariant bundle maps and thus parallel for any Weyl connection. Thus we conclude that

$$\nabla^\sigma \det(\mathsf{P}^\sigma) = (n-1)C\left(\epsilon \otimes \epsilon \otimes (\mathsf{P}^\sigma)^{\otimes^{n-1}} \otimes \nabla^\sigma \mathsf{P}^\sigma\right).$$

Here we have used that the contraction is symmetric in the bilinear forms we enter and in the right hand side the form index of ∇^σ remains uncontracted. Since we assume that P^σ is invertible, linear algebra tells us that contracting two copies of ϵ with $(\mathsf{P}^\sigma)^{\otimes^{n-1}}$ gives $\det(\mathsf{P}^\sigma)\Phi$, where $\Phi \in \Gamma(S^2TM)$ is the inverse of P^σ. Returning to the abstract index notation used in Theorem 3.13 and writing ∇ and P instead of ∇^σ and P^σ, we conclude that $\nabla \det(\mathsf{P}) = 0$ is equivalent to $0 = \mathsf{P}^{ab}\nabla_i \mathsf{P}_{ab}$.

On the other hand, we know the second fundamental form of $s(M) \subset A$ from Theorem 3.13. To determine the mean curvature, we have to contract an inverse metric into this expression, and we already know that this inverse metric is just P^{ij}. Vanishing or non-vanishing of the result is independent of the final contraction with P^{kc}. Thus we conclude that $s(M) \subset A$ is minimal if and only if

$$0 = \mathsf{P}^{ab}(2\nabla_a \mathsf{P}_{bi} - \nabla_i \mathsf{P}_{ab}). \tag{7}$$

In the proof of Theorem 3.13, we have also noted that the Cotton-York tensor is given by the alternation of $\nabla_i \mathsf{P}_{jk}$ in the first two indices. It is well known that this vanishes for projectively flat structures (see [2]) and hence in the projectively flat case, $\nabla_i \mathsf{P}_{jk}$ is completely symmetric. Using this, the claim in the projectively flat case follows immediately.

In the non-flat case, we first have to determine the map φ from Lemma 4 of [18]. In our notation, the map P used there is given by $P|_{L^\pm} = \pm \mathrm{id}$. Using this, the beginning of the proof of Theorem 3.12 readily shows that, in the notation used there, for $\xi \in T_xM \cong L^-_{s(x)}$, we get $\varphi(\xi^i) = (\xi^i, -\mathsf{P}_{ja}\xi^a)$. Now we can combine this with the formula for II_D from Theorem 3.13, which describes the operator $\widehat{A}$ used in [18]. This easily shows that, up to a non-zero factor, that the operator $\widehat{h}$ from [18] is given by

$$\widehat{h}(\xi^i, \eta^j, \zeta^k) = \xi^i\eta^j\zeta^k \mathsf{P}_{ia}\mathsf{P}^{ab}\nabla^s_j \mathsf{P}_{kb}.$$

By definition, the mean curvature form $\widehat{H}$ from [18] is the trace over the first and third entry of this. Thus we have to contract $\widehat{h}_{ijk}$ with P^{ik}, which again leads to $\mathsf{P}^{ab}\nabla^s_j \mathsf{P}_{ab}$. □

Remark 4.5. In the special case of a two-dimensional projective structure the minimality condition (7) was previously obtained in [28, Theorem 4.4].

A deep relation between solutions of the projective Monge-Ampère equation and properly convex projective structures was established in the works [25] by Labourie and [26] by Loftin. Recall that a projective manifold $(M, [\nabla])$ is called *properly convex* if it arises as a quotient of a properly convex open set $\tilde{M} \subset \mathbb{RP}^n$ by a group Γ of projective transformations which acts discretely and properly discontinuously. The projective line segments contained in $\tilde{M}$ project to M to become the geodesics of $[\nabla]$. Therefore, locally, the geodesics of a properly convex projective structure $[\nabla]$ can be mapped diffeomorphically to segments of straight lines, that is, $[\nabla]$ is locally flat. Combining the work of Labourie and Loftin with Theorem 4.4, we obtain:

Corollary 4.6. *Let $(M,[\nabla])$ be a closed oriented locally flat projective manifold. Then $[\nabla]$ is properly convex if and only if $[\nabla]$ arises from a minimal Lagrangian Weyl structure whose Rho tensor is positive definite.*

Proof. Suppose that the flat projective structure $[\nabla]$ arises from a minimal Lagrangian Weyl structure s whose Rho tensor P^s is positive definite. Since s is Lagrangian, P^s is a constant negative multiple of the Ricci tensor of the Weyl connection ∇^s, see [2], and taking into account the sign issue mentioned in Section 2.6. By Theorem 3.13, the nowhere vanishing density $\det(\mathsf{P}^s)$ is preserved by ∇^s, whence an appropriate power defines a volume density that is parallel for ∇^s. Finally, projective flatness implies that the pair (∇^s,P^s) satisfies the hypothesis of Theorem 3.2.1 of [25], which then implies that $[\nabla]$ is properly convex.

Conversely, suppose that $[\nabla]$ is properly convex. By [26, Theorem 4] there is a solution σ to the projective Monge-Ampère equation with right hand side $(-1)^{n+2}\sigma^{n+2}$ and such that σ is negative for the natural orientation on $\mathcal{E}(1)$. By Theorem 4.4, $[\nabla]$ arises from a minimal Lagrangian Weyl structure. Since the Hessian of σ is positive definite, so is the Rho tensor. □

Remark 4.7. Existence and uniqueness of minimal Lagrangian Weyl structures for a given torsion-free AHS structure is an interesting fully non-linear PDE problem. In the special case of projective surfaces, some partial results regarding uniqueness have been obtained in [27] and [30]. See also [31] for a connection to dynamical systems and [29] for a related variational problem on the space of conformal structures.

4.4. Invariant non-linear PDE for other AHS structures

We conclude this article with some remarks on analogs of the projective Monge-Ampère equation for other AHS structures. The first observation is that a small representation theoretic condition is sufficient to obtain an analog of the projectively invariant Hessian, which again is closely related to the Rho tensor.

To formulate this, we need a bit of background. Suppose that (G,P) corresponds to a $|1|$-grading of G and let $G_0\subset P$ be the subgroup determined by the grading. Then this naturally acts on each $\mathfrak{g}_i$, and there is an induced representation on $S^2\mathfrak{g}_1$. We can decompose this representation into irreducibles and there is a unique component whose highest weight is twice the highest weight of $\mathfrak{g}_1$. This is called the *Cartan square* of $\mathfrak{g}_1$ and denoted by $\odot^2\mathfrak{g}_1$. It comes with a canonical G_0-equivariant projection $\pi:\otimes^2\mathfrak{g}_1\to\odot^2\mathfrak{g}_1$. For any parabolic geometry of type (G,P), this induces a natural subbundle $\odot^2T^*M\subset S^2T^*M$ and a natural bundle map $\pi:\otimes^2T^*M\to\odot^2T^*M$.

Proposition 4.8. *Suppose that (G,P) corresponds to a $|1|$-grading on the simple Lie algebra $\mathfrak{g}$ of G. Suppose further, that there is a representation $\mathbb{V}$ of G satisfying the assumptions of Theorem 4.2 whose complexification is a fundamental representation of the complexification of $\mathfrak{g}$, and let $\mathcal{E}M$ denote the natural line bundle induced by $\mathbb{V}/\mathbb{V}^1$.*

(1) There is an invariant differential operator $H : \Gamma(\mathcal{E}M) \to \Gamma(\odot^2 T^*M \otimes \mathcal{E}M)$ of second order.

(2) For a nowhere vanishing section $\sigma \in \Gamma(\mathcal{E}M)$, let P^σ be the Rho tensor of the Weyl structure determined by σ. Then $H(\sigma)$ is a non-zero multiple of $\pi(P^\sigma)\sigma$, where π is the projection to the Cartan square.

Proof. (1) The representation $\mathbb{V}$ induces a tractor bundle on parabolic geometries of type (G, P) to which the construction of BGG sequences can be applied. The first operator H in the resulting sequence is defined on $\Gamma(\mathcal{E}M)$. Now the condition that $\dim(\mathbb{V}/\mathbb{V}^1) = 1$ implies that the complexification of $\mathbb{V}$ is the fundamental representation corresponding to the simple root that induces the $|1|$-grading that defines $\mathfrak{p}$. Since the complexification of $\mathbb{V}$ is a fundamental representation, the results of [5] show that the first operator in the BGG sequence has order two and the target space claimed in (1).

(2) It is also known in general (see [14] or [10]) how to write out H in terms of a Weyl structure with Weyl connection ∇^s and Rho tensor P^s: For $\sigma \in \Gamma(\mathcal{E}M)$, one then has to form $\nabla^2\sigma - \mathsf{P}^s\sigma$, symmetrize and then project to the Cartan square. But if s is the Weyl structure determined by σ, then by definition $\nabla^s\sigma = 0$ and $\mathsf{P}^s = \mathsf{P}^\sigma$, which implies the claim. □

Observe that Theorem 4.3 provides representations $\mathbb{V}$ that satisfy the assumptions of the proposition for conformal and for almost Grassmannian structures. Hence for these two geometries an invariant Hessian is available. It is worth mentioning that, for conformal structures, $\pi(\mathsf{P}^\sigma)$ is the trace-free part of P^σ.

It is also a general fact that the top-exterior power of T^*M is isomorphic to a positive, integral power of the dual $\mathcal{E}^*M$ of $\mathcal{E}M$: By definition, the grading element E acts by multiplication by $\dim(\mathfrak{g}_1)$ on the top exterior power of $\mathfrak{g}_1$, which represents the top exterior power of T^*M. On the other hand, the construction implies that a generator of $\mathbb{V}_0 \subset \mathbb{V}$ will be a lowest weight vector of the complexification of $\mathbb{V}$, so E acts by a negative number on this. The fact that we deal with a fundamental representation implies that $\dim(\mathfrak{g}_{-1})$ is an integral multiple of that number. As in the projective case, this can be phrased as the existence of a tautological section, which can then be used together with copies of $H(\sigma)$ to obtain a section of a line bundle, which can be trivial, a tensor power of $\mathcal{E}$ or a tensor power of $\mathcal{E}^*$. In any case, a nowhere vanishing section of $\mathcal{E}$ determines a canonical section of that bundle (which is the constant 1 in the trivial case), so there is an invariant version of the Monge-Ampère equation. In view of part (2) of Theorem 4.8, for these equations there is always an analog of part (1) of Theorem 4.4.

For some of the structures, there are additional natural sections that can be used together with powers of $H(\sigma)$ to construct other non-linear invariant operators, for example, the conformal metric for conformal structures and partial (density valued) volume forms for Grassmannian structures. Again part (2) of Theorem 4.8 shows that all these equations can be phrased as equations on P^σ, so there should be a relation to submanifold geometry of

Weyl structures in the style of part (2) of Theorem 4.4. All this will be taken up in detail elsewhere.

References

[1] **U. Bader**, **C. Frances**, **K. Melnick**, An embedding theorem for automorphism groups of Cartan geometries, *Geom. Funct. Anal.* **19** (2009), 333–355. 24

[2] **T. N. Bailey**, **M. G. Eastwood**, **A. R. Gover**, Thomas's structure bundle for conformal, projective and related structures, *Rocky Mountain J. Math.* **24** (1994), 1191–1217. 27, 31, 34, 35, 36

[3] **R. J. Baston**, Almost Hermitian symmetric manifolds. I. Local twistor theory, *Duke Math. J.* **63** (1991), 81–112. 3

[4] **R. J. Baston**, Almost Hermitian symmetric manifolds. II. Differential invariants, *Duke Math. J.* **63** (1991), 113–138. 3

[5] **T. Branson**, **A. Čap**, **M. Eastwood**, **A. R. Gover**, Prolongations of geometric overdetermined systems, *Internat. J. Math.* **17** (2006), 641–664. arXiv:math/0402100 31, 37

[6] **R. L. Bryant**, Bochner-Kähler metrics, *J. Amer. Math. Soc.* **14** (2001), 623–715. arXiv:math/0003099 24

[7] **E. Calabi**, Complete affine hyperspheres. I, Symposia Mathematica, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971), (1972), 19–38. 4

[8] **D. M. J. Calderbank**, Selfdual 4-manifolds, projective surfaces, and the Dunajski-West construction, *SIGMA Symmetry Integrability Geom. Methods Appl.* **10** (2014), Paper 035, 18. arXiv:math/0606754 24

[9] **D. M. J. Calderbank**, **T. Diemer**, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, *J. Reine Angew. Math.* **537** (2001), 67–103. arXiv:math/0001158 28

[10] **D. M. J. Calderbank**, **T. Diemer**, **V. Souček**, Ricci-corrected derivatives and invariant differential operators, *Differential Geom. Appl.* **23** (2005), 149–175. arXiv:math/0310311 14, 37

[11] **A. Čap**, **A. R. Gover**, **M. Hammerl**, Holonomy reductions of Cartan geometries and curved orbit decompositions, *Duke Math. J.* **163** (2014), 1035–1070. arXiv:1103.4497 27

[12] **A. Čap**, **A. R. Gover**, **H. R. Macbeth**, Einstein metrics in projective geometry, *Geom. Dedicata* **168** (2014), 235–244. arXiv:1207.0128 27

[13] **A. Čap**, **A. R. Gover**, Projective compactifications and Einstein metrics, *J. Reine Angew. Math.* **717** (2016), 47–75. arXiv:1304.1869 33, 34

[14] **A. Čap**, **J. Slovák**, **V. Souček**, Invariant operators on manifolds with almost Hermitian symmetric structures. I. Invariant differentiation, *Acta Math. Univ. Comenian. (N.S.)* **66** (1997), 33–69. 3, 37

[15] **A. Čap**, **J. Slovák**, Weyl structures for parabolic geometries, *Math. Scand.* **93** (2003), 53–90. 2, 8, 15, 25

[16] **A. Čap**, **J. Slovák**, *Parabolic geometries. I*, *Mathematical Surveys and Monographs* **154**, American Mathematical Society, Providence, RI, 2009, Background and general theory. 1, 2, 5, 6, 7, 9, 11, 13, 14, 15, 16, 18, 19, 24, 25, 27, 32

[17] **A. Čap**, **J. Slovák**, **V. Souček**, Bernstein-Gelfand-Gelfand sequences, *Ann. of Math.* **154** (2001), 97–113. 28

[18] **M. Chursin**, **L. Schäfer**, **K. Smoczyk**, Mean curvature flow of space-like Lagrangian submanifolds in almost para-Kähler manifolds, *Calc. Var. Partial Differential Equations* **41** (2011), 111–125. 34, 35

[19] **M. Dunajski**, **T. Mettler**, Gauge theory on projective surfaces and anti-self-dual Einstein metrics in dimension four, *J. Geom. Anal.* **28** (2018), 2780–2811. arXiv:1509.04276 2, 4, 24, 30, 31

[20] **C. Frances**, Sur le groupe d'automorphismes des géométries paraboliques de rang 1, *Ann. Sci. École Norm. Sup. (4)* **40** (2007), 741–764. 24

[21] **M. Herzlich**, Parabolic geodesics as parallel curves in parabolic geometries, *Internat. J. Math.* **24** (2013), 1350067, 16. arXiv:1207.4005 3, 8, 10

[22] **R. Hildebrand**, The cross-ratio manifold: a model of centro-affine geometry, *Int. Electron. J. Geom.* **4** (2011), 32–62. 5

[23] **R. Hildebrand**, Half-dimensional immersions in para-Kähler manifolds, *Int. Electron. J. Geom.* **4** (2011), 85–113. 5

[24] **S. Kobayashi**, **T. Nagano**, On filtered lie algebras and geometric structures. I, *J. Math. Mech.* **13** (1964), 875–907. 7, 20

[25] **F. Labourie**, Flat projective structures on surfaces and cubic holomorphic differentials, *Pure Appl. Math. Q.* **3** (2007), 1057–1099. arXiv:math/0611250 35, 36

[26] **J. C. Loftin**, Affine spheres and convex $\mathbb{RP}^n$-manifolds, *Amer. J. Math.* **123** (2001), 255–274. 5, 35, 36

[27] **T. Mettler**, Geodesic rigidity of conformal connections on surfaces, *Math. Z.* **281** (2015), 379–393. arXiv:1410.8501 36

[28] **T. Mettler**, Minimal Lagrangian connections on compact surfaces, *Adv. Math.* **354** (2019). arXiv:1609.08033 2, 4, 25, 35

[29] **T. Mettler**, Extremal conformal structures on projective surfaces, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **XX** (2020), 1621–1663. arXiv:1510.01043 36

[30] **T. Mettler**, **G. P. Paternain**, Convex projective surfaces with compatible Weyl connection are hyperbolic, 2018, to appear in *Anal. PDE*. arXiv:1804.04616 36

[31] **T. Mettler**, **G. P. Paternain**, Holomorphic differentials, thermostats and Anosov flows, *Math. Ann.* **373** (2019), 553–580. arXiv:1706.03554 36

[32] **A. Wienhard**, An invitation to higher Teichmüller theory, 2018. arXiv:1803.06870 2

A.Č. – Faculty of Mathematics, University of Vienna, Oskar–Morgenstern–Platz 1, 1090 Wien, Austria

Email address: `Andreas.Cap@univie.ac.at`

T.M. – Faculty of Mathematics, UniDistance Suisse, Schinerstrasse 18, 3900 Brig, Switzerland

Email address: `thomas.mettler@fernuni.ch`