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Metrisability of Projective Surfaces and Pseudo-Holomorphic Curves

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ABSTRACT. We show that the metrisability of an oriented projective surface is equivalent to the existence of pseudo-holomorphic curves. A projective structure $\mathfrak{p}$ and a volume form σ on an oriented surface M equip the total space of a certain disk bundle $Z \to M$ with a pair $(J_{\mathfrak{p}}, \mathfrak{J}_{\mathfrak{p},\sigma})$ of almost complex structures. A conformal structure on M corresponds to a section of $Z \to M$ and $\mathfrak{p}$ is metrisable by the metric g if and only if $[g] : M \to Z$ is a pseudo-holomorphic curve with respect to $J_{\mathfrak{p}}$ and $\mathfrak{J}_{\mathfrak{p},dA_g}$.

1. Introduction

A *projective structure* on a smooth manifold consists of an equivalence class $\mathfrak{p}$ of torsion-free connections on its tangent bundle, where two such connections are called equivalent if they have the same geodesics up to parametrisation. A projective structure $\mathfrak{p}$ is called *metrisable* if it contains the Levi-Civita connection of some Riemannian metric. The problem of (locally) characterising the projective structures that are metrisable was first studied in the work of R. Liouville [17] in 1889, but was solved only relatively recently by Bryant, Dunajski and Eastwood for the case of two dimensions [2]. Since then, there has been renewed interest in the problem, see [5, 6, 8, 10, 11, 13, 14, 25, 27] for related recent work.

The purpose of this short note is to show that in the case of an oriented projective surface $(M, \mathfrak{p})$, the metrisability of $\mathfrak{p}$ is equivalent to the existence of certain pseudo-holomorphic curves.

An orientation compatible complex structure on M corresponds to a section of the bundle $\pi : Z \to M$ whose fibre at $x \in M$ consists of the orientation compatible linear complex structures on T_xM. The choice of a torsion-free connection ∇ on TM equips Z with an almost complex structure J [7, 26]. Namely, at $j \in Z$ we lift j horizontally and take a natural complex structure on each fibre vertically. It turns out that J is always integrable and does only depend on the projective equivalence class $\mathfrak{p}$ of ∇, we thus denote it by $J_{\mathfrak{p}}$. Reversing the orientation on each fibre yields another almost complex structure $\mathfrak{J}$ which is however never integrable and is not projectively invariant. Fixing a volume form σ on the projective surface $(M, \mathfrak{p})$ determines a unique representative connection ${}^{\sigma}\nabla \in \mathfrak{p}$ which preserves σ. We will write $\mathfrak{J}_{\mathfrak{p},\sigma}$ for the non-integrable almost complex structure arising from ${}^{\sigma}\nabla \in \mathfrak{p}$.

The choice of a conformal structure $[g]$ on an oriented surface M defines an orientation compatible complex structure by rotating a tangent vector counter-clockwise by $\pi/2$ with respect to $[g]$. Thus, we may think of a conformal structure

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as a section $[g] : M \to Z$. Denoting the area form of a Riemannian metric g by dA_g, we show:

Theorem 1.1. *An oriented projective surface $(M, \mathfrak{p})$ is metrisable by the metric g on M if and only if $[g] : M \to (Z, J_{\mathfrak{p}})$ is a holomorphic curve and $[g] : M \to (Z, \mathfrak{J}_{\mathfrak{p},dA_g})$ is a pseudo-holomorphic curve.*

Applying a general existence result for pseudo-holomorphic curves [24, Theorem III] it follows that locally we can always find a Riemannian metric g so that $[g] : M \to (Z, J_{\mathfrak{p}})$ is a holomorphic curve or so that $[g] : M \to (Z, \mathfrak{J}_{\mathfrak{p},dA_g})$ is a pseudo-holomorphic curve. The geometric significance of the existence of such (pseudo-)holomorphic curves is given in Proposition 2.8 below.

The construction of the (integrable) almost complex structure $J_{\mathfrak{p}}$ on Z given in [7, 26] is adapted from the construction of an almost complex structure J on the twistor space $Y \to N$ of an oriented Riemannian 4-manifold (N, g), see [1]. In the Riemannian setting the almost complex structure J is integrable if and only if g is self-dual. In [12], Eells–Salamon observe that reversing the orientation on each fibre of $Y \to N$ associates another almost complex structure $\mathfrak{J}$ on Y to (N, g) which is never integrable. Thus, the non-integrable almost complex structure $\mathfrak{J}$ used here may be thought of as the affine analogue of the non-integrable almost complex structure in oriented Riemannian 4-manifold geometry.

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2. Pseudo-Holomorphic Curves and Metrisability

Recall that the set of torsion-free connections on the tangent bundle of a surface M is an affine space modelled on the smooth sections of the vector bundle $V = S^2(T^*M) \otimes TM$. We have a natural trace mapping $\mathrm{tr} : V \to T^*M$, given in abstract index notation by $A^i_{jk} \mapsto A^k_{ik}$, as well as an inclusion $\mathrm{Sym} : T^*M \to V$, given by $b_i \mapsto \delta^i_j b_k + \delta^i_k b_j$. The bundle V thus decomposes as $V = V_0 \oplus T^*M$, where V_0 denotes the trace-free part of V. We have (Cartan, Eisenhart, Weyl) – the reader may also consult [9] for a modern reference:

Lemma 2.1. *Two torsion-free connections ∇ and ∇' on TM are projectively equivalent if and only if there exists a 1-form ξ on M so that $\nabla - \nabla' = \mathrm{Sym}(\xi)$.*

This gives immediately:

Lemma 2.2. *Let $(M, \mathfrak{p})$ be an oriented projective surface and σ a volume form on M. Then there exists a unique representative connection ${}^{\sigma}\nabla \in \mathfrak{p}$ preserving σ.*

Proof. Let $\nabla \in \mathfrak{p}$ be a representative connection. Since σ is a volume form there exists a unique 1-form α on M such that $\nabla\sigma = \alpha \otimes \sigma$. An elementary computation shows that the connection $\nabla + \mathrm{Sym}(\xi)$ satisfies

$$(\nabla + \mathrm{Sym}(\xi))\,\sigma = \nabla\sigma - 3\xi \otimes \sigma,$$

for all $\xi \in \Omega^1(M)$. Thus the connection ${}^{\sigma}\nabla = \nabla + \frac{1}{3}\mathrm{Sym}(\alpha)$ preserves σ and clearly is the only connection in $\mathfrak{p}$ doing so. □

We also have:

Lemma 2.3. *Let $\varphi \in \Gamma(V_0)$ and ∇ be a torsion-free connection on TM. Then $\nabla + \varphi$ preserves a volume form σ on M if and only if ∇ preserves the volume form σ.*

Proof. Since $\varphi \in \Gamma(V_0)$, an elementary computation shows that the connections ∇ and $\nabla + \varphi$ induce the same connection on the bundle $\Lambda^2(T^*M)$ whose non-vanishing sections are the volume forms. □

For our purposes it is convenient to construct the almost complex structures $(J, \mathfrak{J})$ associated to ∇ in terms of the connection form θ on the oriented frame bundle of M. The oriented frame bundle F of the oriented surface M is the bundle $\upsilon : F \to M$ whose fibre at $x \in M$ consists of the linear isomorphisms $u : \mathbb{R}^2 \to T_xM$ that are orientation preserving with respect to the standard orientation on $\mathbb{R}^2$ and the given orientation on T_xM. The group $\mathrm{GL}^+(2,\mathbb{R})$ acts transitively from the right on each fibre by the rule $R_a(u) = u \circ a$ for all $a \in \mathrm{GL}^+(2,\mathbb{R})$, $u \in F$ and this action turns $\upsilon : F \to M$ into a principal right $\mathrm{GL}^+(2,\mathbb{R})$-bundle. The total space F carries a tautological $\mathbb{R}^2$-valued 1-form ω defined by $\omega_u = u^{-1} \circ \upsilon'_u$ and ω satisfies the equivariance property

$$R_a^*\omega = a^{-1}\omega \tag{2.1}$$

for all $a \in \mathrm{GL}^+(2,\mathbb{R})$. We may embed $\mathrm{GL}(1,\mathbb{C})$ as the subgroup of $\mathrm{GL}^+(2,\mathbb{R})$ consisting of matrices that commute with the standard linear complex structure on $\mathbb{R}^2$. Note that may think of the oriented frame bundle $\upsilon : F \to M$ as a principal $\mathrm{GL}(1,\mathbb{C})$-bundle over $Z = F/\mathrm{GL}(1,\mathbb{C})$. We may describe an almost complex structure on Z by describing the pullback of its (1,0)-forms to F. The pullback of a 1-form on Z to F is *semi-basic* for the projection $\nu : F \to Z$, that is, it vanishes when evaluated on vector fields that are tangent to the fibres of ν. For $y \in \mathfrak{gl}(2,\mathbb{R})$ we denote by Y_y the vector field on F that is generated by the flow $R_{\exp(ty)}$. Clearly, the vector fields Y_y for $y \in \mathfrak{gl}(1,\mathbb{C})$ span the vector fields on F that are tangent to the fibres of ν.

Let ∇ be a torsion-free connection on TM with connection form $\theta = (\theta^i_j)$ on F. Recall that θ satisfies the equivariance property

$$R_a^*\theta = a^{-1}\theta a \tag{2.2}$$

for all $a \in \mathrm{GL}^+(2,\mathbb{R})$ and the structure equations

$$\begin{aligned} \mathrm{d}\omega^i &= -\theta^i_j \wedge \omega^j, \\ \mathrm{d}\theta^i_j &= -\theta^i_k \wedge \theta^k_j + \Theta^i_j, \end{aligned} \tag{2.3}$$

where $\Theta = (\Theta^i_j)$ denotes the curvature form of θ. Since θ is a principal connection on F it also satisfies $\theta(Y_y) = y$ for all $y \in \mathfrak{gl}(2,\mathbb{R})$. Since the Lie algebra of

$\mathrm{GL}(1,\mathbb{C})$ is spanned by the matrices of the form

$$\begin{pmatrix} z & -w \\ w & z \end{pmatrix}$$

for $(z,w) \in \mathbb{R}^2$, the complex-valued 1-forms on F that are semi-basic for the projection $\nu : F \to Z$ are spanned by the forms $\omega = \omega^1 + \mathrm{i}\omega^2$ and

$$\zeta = (\theta^1_1 - \theta^2_2) + \mathrm{i}\left(\theta^1_2 + \theta^2_1\right)$$

and their complex conjugates. We now have:

Proposition 2.4. *Let ∇ be a torsion-free connection on TM with connection form $\theta = (\theta^i_j)$ on F. Then there exists a unique pair $(J,\mathfrak{J})$ of almost complex structures on Z whose* (1,0)*-forms pull back to become linear combinations of the forms (ω,ζ) in the case of J and to $(\omega,\overline{\zeta})$ in the case of $\mathfrak{J}$. Moreover, the almost complex structure J is always integrable, whereas $\mathfrak{J}$ is never integrable.*

Proof. Writing

$$r\mathrm{e}^{\mathrm{i}\phi} \simeq \begin{pmatrix} r\cos\phi & -r\sin\phi \\ r\sin\phi & r\cos\phi \end{pmatrix}$$

for the elements of $\mathrm{GL}(1,\mathbb{C})$, the equivariance property (2.1) of ω and (2.2) of θ implies

$$(R_{r\mathrm{e}^{\mathrm{i}\phi}})^*\omega = \frac{1}{r}\mathrm{e}^{\mathrm{i}\phi}\omega \quad \text{and} \quad (R_{r\mathrm{e}^{\mathrm{i}\phi}})^*\zeta = \mathrm{e}^{-2\mathrm{i}\phi}\zeta. \tag{2.4}$$

It follows that there exists a unique almost complex structure J on Z whose (1,0)-forms pull back to F to become linear combinations of the forms ω,ζ. Likewise there exists a unique almost complex structure $\mathfrak{J}$ on Z whose (1,0)-forms pull back to F to become linear combinations of the forms $\omega,\overline{\zeta}$. Furthermore, simple computations using the structure equations (2.3) imply that

$$0 = \mathrm{d}\zeta\wedge\omega\wedge\zeta = \mathrm{d}\omega\wedge\omega\wedge\zeta.$$

Consequently, the Newlander-Nirenberg theorem [23] implies that J is integrable. On the other hand, we get

$$\mathrm{d}\omega\wedge\omega\wedge\overline{\zeta} = \frac{1}{2}\omega\wedge\overline{\omega}\wedge\zeta\wedge\overline{\zeta}$$

so that $\mathfrak{J}$ is never integrable. □

Remark 2.5. The equivariance properties (2.4) imply that the bundles

$$H = \nu'\{\mathrm{Re}(\zeta)=0, \mathrm{Im}(\zeta)=0\} \quad \text{and} \quad V = \nu'\{\mathrm{Re}(\omega)=0, \mathrm{Im}(\omega)=0\}$$

are well-defined distributions on Z that are invariant with respect to J (and $\mathfrak{J}$). Hence we have $TZ = H \oplus V$.

For the convenience of the reader, we also show [7, 26]:

Proposition 2.6. *Suppose the torsion-free connections ∇ and ∇' on TM are projectively equivalent, then they induce the same integrable almost complex structure J on Z.*

Proof. The connections ∇ and ∇' are projectively equivalent if and only if there exists a 1-form ξ on M such that $\nabla' = \nabla + \mathrm{Sym}(\xi)$. Writing $\theta = (\theta^i_j)$ for the connection form of ∇ on F and $\upsilon^*\xi = x_i\omega^i$ for real-valued functions x_i on F, the connection form θ' of ∇' becomes

$$\theta' = \theta + \begin{pmatrix} 2x_1\omega^1 + x_2\omega^2 & x_2\omega^1 \\ x_1\omega^2 & x_1\omega^1 + 2x_2\omega^2 \end{pmatrix}.$$

Consequently, we obtain

$$\zeta' = \zeta + (x_1\omega^1 - x_2\omega^2) + \mathrm{i}(x_2\omega^1 + x_1\omega^2) = \zeta + (x_1 + \mathrm{i}x_2)\omega$$

which shows that the complex span of ω, ζ is the same as the one of ω, ζ' and hence the two integrable almost complex structures are the same. □

Remark 2.7. For a projective structure $\mathfrak{p}$ on M we will write $J_{\mathfrak{p}}$ for the integrable almost complex structure defined by any representative connection $\nabla \in \mathfrak{p}$. For a projective structure $\mathfrak{p}$ and a volume form σ on M we will write $\mathfrak{J}_{\mathfrak{p},\sigma}$ for the non-integrable almost complex structure defined by the representative connection ${}^{\sigma}\nabla \in \mathfrak{p}$. Note that the non-integrable almost complex structure is not projectively invariant.

Recall that a Weyl connection for a conformal structure $[g]$ is a torsion-free connection ${}^{[g]}\nabla$ on TM which preserves $[g]$. Fixing a Riemannian metric $g \in [g]$, the Weyl connections for $[g]$ can be written as ${}^{[g]}\nabla = {}^{g}\nabla + g \otimes B - \mathrm{Sym}(\beta)$ for some 1-form β on M and where B denotes the g-dual vector field to β. In [20] and in the language of thermostats in [22], it was observed that for every choice of a conformal structure $[g]$ on a projective surface $(M, \mathfrak{p})$, there exists a unique Weyl connection ${}^{[g]}\nabla$ for $[g]$ and a unique 1-form $\varphi \in \Gamma(V_0)$ so that ${}^{[g]}\nabla + \varphi$ is a representative connection of $\mathfrak{p}$. Moreover the endomorphism $\varphi(X)$ is symmetric with respect to $[g]$ for every vector field X on M. We call ${}^{[g]}\nabla$ the Weyl connection determined by $[g]$. Explicitly, if ∇ is any representative connection of $\mathfrak{p}$, $g \in [g]$ and if we define a vector field $B = \frac{3}{4}\mathrm{tr}\left(g^{\sharp} \otimes (\nabla - {}^{g}\nabla)_0\right)$, then

$$\varphi = \left(\nabla - {}^{g}\nabla - g \otimes B\right)_0 \qquad \text{and} \qquad {}^{[g]}\nabla = {}^{g}\nabla + g \otimes B - \mathrm{Sym}(\beta),$$

where A_0 denotes the trace-free part of a tensor field $A \in \Gamma(S^2(T^*M) \otimes TM)$. We refer the reader to [20, 22] for a proof that ${}^{[g]}\nabla$ and φ do satisfy the claimed properties.

Proposition 2.8. *Let $(M, \mathfrak{p})$ be an oriented projective surface and g a Riemannian metric on M. Then we have:*

(i) $\mathfrak{p}$ *contains a Weyl connection for* $[g]$ *if and only if* $[g] : M \to (Z, J_{\mathfrak{p}})$ *is a holomorphic curve;*

(ii) *the Weyl connection determined by* $[g]$ *is the Levi-Civita connection of* g *if and only if* $[g] : M \to (Z, \mathfrak{J}_{\mathfrak{p},dA_g})$ *is a pseudo-holomorphic curve.*

Remark 2.9. Here we say $[g] : M \to (Z, \mathfrak{J})$ is a (pseudo-)holomorphic curve if the image $\Sigma = [g](M) \subset Z$ admits the structure of a (pseudo-)holomorphic curve. By admitting the structure of (pseudo-)holomorphic curve, we mean that Σ can be equipped with a complex structure J, so that the inclusion $\iota : \Sigma \to Z$ is $(J, \mathfrak{J})$-linear, that is, satisfies $\mathfrak{J} \circ \iota' = \iota' \circ J$.

As an immediate consequence, we obtain the Theorem 1.1:

Proof of Theorem 1.1. The projective structure $\mathfrak{p}$ is metrisable by g if and only if the Weyl connection determined by $[g]$ is the Levi-Civita connection of g and the 1-form φ vanishes identically. The claim follows by applying Proposition 2.8. □

For the proof of Proposition 2.8 we also need the following Lemma:

Lemma 2.10. *Let $(Z, \mathfrak{J})$ be an almost complex four-manifold and $\omega, \chi \in \Omega^1(Z, \mathbb{C})$ a basis for the (1,0)-forms of Z. Suppose $\iota : \Sigma \to Z$ is an immersed surface so that $\iota^*(\omega \wedge \overline{\omega})$ is non-vanishing on Σ. Then Σ admits the structure of a pseudo-holomorphic curve if and only if $\iota^*(\omega \wedge \chi)$ vanishes identically on Σ.*

Proof. Since $\iota^*(\omega \wedge \overline{\omega})$ is non-vanishing on Σ, the forms $\iota^*\omega$ and $\iota^*\overline{\omega}$ span the complex-valued 1-forms on Σ. Recall that $\iota : \Sigma \to Z$ is $(j, \mathfrak{J})$-linear if and only if the pullback of every (1,0)-form on Z is a (1,0)-form on Σ, the claim follows. □

Proof of Proposition 2.8. Let g be a Riemannian metric on the oriented projective surface $(M, \mathfrak{p})$. Without losing generality we can assume that the projective structure $\mathfrak{p}$ arises from a connection of the form ${}^{[g]}\nabla + \varphi$. The Weyl connection ${}^{[g]}\nabla$ satisfies

$$
{}^{[g]}\nabla dA_g = 2\beta \otimes dA_g
$$

for some 1-form β on M and hence can be written as ${}^{[g]}\nabla = {}^{g}\nabla + g \otimes \beta^\sharp - \mathrm{Sym}(\beta)$.

Now suppose $\nabla \in \mathfrak{p}$ preserves the volume form dA_g of g. Then, by Lemma 2.3 it must be of the form

$$
\nabla = {}^{[g]}\nabla + \varphi + \frac{2}{3}\mathrm{Sym}(\beta) = {}^{g}\nabla + g \otimes \beta^\sharp - \frac{1}{3}\mathrm{Sym}(\beta) + \varphi. \tag{2.5}
$$

Proposition 2.4 and Lemma 2.10 imply that the condition that $[g] : M \to Z$ defines a pseudo-holomorphic curve with respect to $J_{\mathfrak{p}}$ respectively $\mathfrak{J}_{\mathfrak{p}, dA_g}$ is equivalent to the condition that on the pullback bundle $[g]^*F \to M$ the form $\omega \wedge \zeta$, respectively $\omega \wedge \overline{\zeta}$ vanishes identically, where ζ is computed from the connection form of ∇ and where we think of F as fibering over Z. Keeping this in mind we now compute the pullback of the forms ζ and $\overline{\zeta}$ to $[g]^*F$. Recall that the semi-basic 1-forms on F are spanned by the components of ω, hence there exist unique real-valued functions $g_{ij} = g_{ji}$ on F so that $\upsilon^* g = g_{ij}\omega^i \otimes \omega^j$. Likewise, there exist unique real-valued functions b_i on F so that $\upsilon^*\beta = b_i\omega^i$ and unique real-valued function $A^i_{jk} = A^i_{kj}$ on F so that $(\upsilon^*\varphi)^i_j = A^i_{jk}\omega^k$. The functions A^i_{jk} satisfy furthermore $A^k_{ki} = 0$ and $g_{ik}A^k_{jl} = g_{jk}A^k_{il}$ since φ takes values in the endomorphisms of TM that are trace-free and symmetric with respect to g. The Levi-Civita connection (ψ^i_j) of g is the unique principal $\mathrm{GL}^+(2, \mathbb{R})$-connection on F that satisfies

$$
\begin{aligned}
\mathrm{d}\omega^i &= -\psi^i_j \wedge \omega^j, \\
\mathrm{d}g_{ij} &= g_{ik}\psi^k_j + g_{kj}\psi^k_i.
\end{aligned}
$$

The pullback bundle $P := [g]^*F$ is cut out by the equations $g_{11} = g_{22}$ and $g_{12} = 0$. On P we have

$$
\begin{aligned}
0 = \mathrm{d}g_{12} &= g_{11}\psi^1_2 + g_{22}\psi^2_1 = g_{11}(\psi^1_2 + \psi^2_1), \\
0 = \mathrm{d}g_{11} - \mathrm{d}g_{22} &= 2g_{11}\psi^1_1 - 2g_{22}\psi^2_2 = g_{11}(\psi^1_1 - \psi^2_2)
\end{aligned}
$$

On P the condition $g_{ik}A^k_{jl} = g_{jk}A^k_{il}$ implies $A^2_{11} = -A^2_{22}$ and $A^1_{22} = -A^1_{11}$. Writing $A^1_{11} = a_1$ and $A^2_{22} = a_2$ and using (2.5), the connection form θ of ∇ thus becomes

$$\theta = \begin{pmatrix} \psi^1_1 & -\psi^2_1 \\ \psi^2_1 & \psi^1_1 \end{pmatrix} + \begin{pmatrix} b_1\omega^1 & b_1\omega^2 \\ b_2\omega^1 & b_2\omega^2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 2b_1\omega^1 + b_2\omega^2 & b_2\omega^1 \\ b_1\omega^2 & b_1\omega^1 + 2b_2\omega^2 \end{pmatrix} + \begin{pmatrix} a_1\omega^1 - a_2\omega^2 & -a_2\omega^1 - a_1\omega^2 \\ -a_2\omega^1 - a_1\omega^2 & -a_1\omega^1 + a_2\omega^2 \end{pmatrix}$$

Introducing the complex notation $a = a_1 + \mathrm{i}a_2$ and $b = \frac{1}{2}(b_1 - \mathrm{i}b_2)$, we obtain from a simple calculation

$$\zeta = (\theta^1_1 - \theta^2_2) + \mathrm{i}(\theta^1_2 + \theta^2_1) = \frac{4}{3}\overline{b}\omega + 2\overline{a\omega},$$

where we write $\omega = \omega^1 + \mathrm{i}\omega^2$.

Finally, since $[g] : M \to (Z, J_{\mathfrak{p}})$ is a holomorphic curve if and only if $\omega \wedge \zeta$ vanishes identically on P, it follows that $[g] : M \to (Z, J_{\mathfrak{p}})$ is a holomorphic curve if and only if

$$0 = \omega \wedge \zeta = 2\overline{a}\omega \wedge \overline{\omega}$$

which is equivalent to φ vanishing identically. This shows (i).

Likewise $[g] : M \to (Z, \mathfrak{J}_{\mathfrak{p},dA_g})$ is a pseudo-holomorphic curve if and only if

$$0 = \omega \wedge \overline{\zeta} = \frac{4}{3}b\omega \wedge \overline{\omega}$$

on P. This is equivalent to β vanishing identically. This shows (ii). $\square$

As a corollary we obtain:

Corollary 2.11. *Let $(M, \mathfrak{p})$ be a projective surface. Then locally $\mathfrak{p}$ contains*

(i) *a Weyl connection ${}^{[g]}\nabla$ for some conformal structure $[g]$;*
(ii) *a connection of the form ${}^{\tilde g}\nabla + \varphi$ for some Riemannian metric $\tilde g$ and some $\varphi \in \Gamma(V_0)$ with φ taking values in the endomorphisms that are $\tilde g$-symmetric.*

Remark 2.12. The first statement of Proposition 2.8 and Corollary 2.11 was previously obtained in [19].

Proof of Corollary 2.11. We first consider the case (ii). We fix a volume form σ on M. We need to show that in a neighbourhood U_x of every point $x \in M$ there exists a conformal structure $[g]$ which is a pseudo-holomorphic curve into the total space of the bundle $\pi : Z \to M$, where we equip Z with the almost complex structure $\mathfrak{J}_{\mathfrak{p},\sigma}$. Choose $j \in Z$ with $\pi(j) = x$. Recall from Remark 2.5 that the subspace $H_j \subset T_jZ$ is invariant under $\mathfrak{J}_{\mathfrak{p},\sigma}$. Now [24, Theorem III] implies that there exists a pseudo-holomorphic curve $\Sigma \subset (Z, \mathfrak{J}_{\mathfrak{p},\sigma})$ which contains j and has H_j as its tangent space at j. Since $H_j \subset T_jZ$ is horizontal, the restriction $\pi'_j|_{H_j} : H_j \to T_xM$ is an isomorphism. Therefore, the restriction of π to Σ is a local diffeomorphism in some neighbourhood of j. Hence there exists a neighbourhood U_x of $x \in M$ and a section $[g] : U_x \to Z$ so that $[g](U_x) \subset \Sigma$. Thus, $[g] : U_x \to (Z, \mathfrak{J}_{\mathfrak{p},\sigma})$ is a pseudo-holomorphic curve in the sense of Remark 2.9. Taking $\tilde g$ to be the unique metric in $[g]$ with volume form σ and applying Proposition 2.8 shows the claim. The case (i) follows in the same fashion, except that [24] is not needed, as $J_{\mathfrak{p}}$ is integrable and hence the construction of a holomorphic curve realising a prescribed $J_{\mathfrak{p}}$-invariant tangent plane is an elementary exercise. $\square$

Remark 2.13. Locally we can always find a holomorphic curve $[g] : M \to (Z, J_{\mathfrak{p}})$, but globally this is not always possible. A properly convex projective structure $\mathfrak{p}$ on a closed surface M with $\chi(M) < 0$ admits a holomorphic curve $[g] : M \to (Z, J_{\mathfrak{p}})$ if and only if $\mathfrak{p}$ is hyperbolic [22]. One would expect that a corresponding global non-existence result should also hold in the pseudo-holomorphic setting for a suitable class of projective surfaces.

Remark 2.14. If $(M, \mathfrak{p})$ is a closed oriented projective surface of with $\chi(M) < 0$, then there exists at most one holomorphic curve $[g] : M \to (Z, J_{\mathfrak{p}})$, see [21].

Remark 2.15. Hitchin [15] gave a twistorial construction of (complex) two-dimensional holomorphic projective structures. In the holomorphic category such a projective structure corresponds to a complex surface Z having a family of rational curves with self-intersection number one. Denoting the canonical bundle of Z by K_Z, such a holomorphic projective surface is metrisable if and only if $K_Z^{-2/3}$ admits a holomorphic section which intersects each rational curve in Z at two points [2, 3, 16].

Remark 2.16. The notion of a projective structure also makes sense in the complex setting and such structures are referred to as *c-projective*, see [4]. Correspondingly, there is a Kähler metrisability problem of c-projective structures. Some obstructions to Kähler metrisability of a (complex) two-dimensional c-projective structure have been obtained in [18].

We conclude by describing the holomorphic curves for the standard projective structure $\mathfrak{p}_0$ on the 2-sphere whose geodesics are the great circles.

Example 2.17. Let S^2 denote the sphere of radius 1 centred at the origin in $\mathbb{R}^3$ and g its induced round metric of constant Gauss curvature 1 whose geodesics are the great circles. We equip S^2 with its standard orientation.

Recall that the unit tangent bundle $\lambda : T_1S^2 \to S^2$ of (S^2, g) carries a canonical coframing $(\omega_1, \omega_2, \psi)$, where ω_1, ω_2 span the 1-forms on T_1S^2 that are semi-basic for the projection λ and ψ denotes the Levi-Civita connection form of g. The 1-forms $(\omega_1, \omega_2, \psi)$ satisfy the structure equations

$$\mathrm{d}\omega_1 = -\omega_2 \wedge \psi \quad \text{and} \quad \mathrm{d}\omega_2 = -\psi \wedge \omega_1 \quad \text{and} \quad \mathrm{d}\psi = -\omega_1 \wedge \omega_2. \tag{2.6}$$

Let $\hat{g}$ be a Riemannian metric on S^2 and write $\lambda^*\hat{g} = \hat{g}_{ij}\omega_i \otimes \omega_j$ for unique real-valued functions $\hat{g}_{ij} = \hat{g}_{ji}$ on T_1S^2. Phrased in modern language (c.f. [2]) and applied to the case of the 2-sphere, R. Liouville's result [17] implies that if the metrics $\hat{g}$ and g have the same unparametrised geodesics then the functions $h_{ij} := \hat{g}_{ij}(\hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}^2)^{-2/3}$ satisfy the linear differential equations

$$\begin{aligned} \mathrm{d}h_{11} &= -2h_1\omega_2 + 2h_{12}\psi, \\ \mathrm{d}h_{12} &= h_1\omega_1 - h_2\omega_2 - (h_{11} - h_{22})\psi, \\ \mathrm{d}h_{22} &= 2h_2\omega_1 - 2h_{12}\psi, \end{aligned} \tag{2.7}$$

for some smooth real-valued functions h_i on T_1S^2. Conversely, a solution to (2.7) on T_1S^2 satisfying $h_{11}h_{22} - h_{12}^2 \neq 0$ gives a Riemannian metric $\hat{g}$ on S^2 with $\lambda^*\hat{g} = (h_{ij}(h_{11}h_{22} - h_{12}^2)^{-2})\omega_i \otimes \omega_j$ and that has the same unparametrised geodesics as g.

Applying the exterior derivative to the above system of equations implies the existence of a unique real-valued function h on T_1S^2 such that

$$\begin{aligned} \mathrm{d}h_1 &= -h_{12}\omega_1 + (h_{11} + h)\omega_2 + h_2\psi, \\ \mathrm{d}h_2 &= -(h_{22} + h)\omega_1 + h_{12}\omega_2 - h_1\psi. \end{aligned}$$

Taking yet another exterior derivative gives that

$$\mathrm{d}h = -2h_1\omega_1 + 2h_2\omega_2.$$

Writing

$$\vartheta = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 \\ \omega_1 & 0 & -\psi \\ \omega_2 & \psi & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h & h_2 & -h_1 \\ h_2 & -h_{22} & h_{12} \\ -h_1 & h_{12} & -h_{11} \end{pmatrix}$$

the above system of differential equations can be expressed as

$$\mathrm{d}H + \vartheta H + H\vartheta^t = 0.$$

The structure equations (2.6) imply that $\mathrm{d}\vartheta + \vartheta \wedge \vartheta = 0$, hence we may write $\vartheta = \Xi^{-1}\mathrm{d}\Xi$ for some diffeomorphism $\Xi : T_1S^2 \to \mathrm{SO}(3)$. It follows that the solutions are of the form $H = \Xi^{-1}C(\Xi^{-1})^t$ for some constant symmetric 3-by-3 matrix C. In particular, taking $C = AA^t$ for some $A \in \mathrm{SL}(3, \mathbb{R})$, we obtain a solution H_A providing a metric $\hat{g}_A$ on S^2 having the great circles as its geodesics.

Finally, in order to construct the holomorphic curve $[\hat{g}_A] : S^2 \to Z$ from H_A, we interpret Z as an associated bundle to T_1S^2. We will only give a sketch of the construction and refer the reader to [22, §4] for additional details. The orientation and metric turn S^2 into a Riemann surface and hence a conformal structure on S^2 is given in terms of a Beltrami differential. Denoting the canonical bundle of S^2 by K_{S^2}, a Beltrami differential is a section μ of $\overline{K_{S^2}} \otimes K_{S^2}^{-1}$ satisfying $|\mu(x)| < 1$ for all $x \in S^2$, where $|\cdot|$ denotes the norm induced by the natural Hermitian bundle metric on $\overline{K_{S^2}} \otimes K_{S^2}^{-1}$. The Riemannian metric g gives an isomorphism $\overline{K_{S^2}} \otimes K_{S^2}^{-1} \simeq K_{S^2}^{-2}$ and thus Z may be identified with $T_1S^2 \times_{S^1} \mathbb{D}$, where S^1 acts by usual rotation on T_1S^2 and by $z \cdot \mathrm{e}^{\mathrm{i}\phi} = z\mathrm{e}^{-2\mathrm{i}\phi}$ on the open unit disk $\mathbb{D} \subset \mathbb{C}$. A holomorphic curve $[\hat{g}] : S^2 \to Z$ is therefore represented by a map $\mu : T_1S^2 \to \mathbb{D}$. Explicitly, the conformal structure arising from a Riemannian metric $\hat{g}$ on S^2 is represented by the map

$$\mu = \frac{p - q + 2\mathrm{i}r}{p + q + 2\sqrt{pq - r^2}},$$

where we write $\lambda^*\hat{g} = p\omega_1 \otimes \omega_1 + 2r\omega_1 \circ \omega_2 + q\omega_2 \otimes \omega_2$ for unique real-valued functions p, q, r on T_1S^2. In our case, the holomorphic curve $[\hat{g}_A] : S^2 \to Z$ is thus represented by μ with

$$p = \frac{h_{11}}{(h_{11}h_{22} - h_{12}^2)^2}, \qquad r = \frac{h_{12}}{(h_{11}h_{22} - h_{12}^2)^2}, \qquad q = \frac{h_{22}}{(h_{11}h_{22} - h_{12}^2)^2}$$

and where the functions h_{ij} arise from H_A as above.

Remark 2.18. In the case of the standard projective structure on S^2 the complex surface $(Z, J_{\mathfrak{p}_0})$ is biholomorphic to $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ and moreover, the image of a holomorphic curve $[g] : S^2 \to Z$ is a smooth quadric, see [19]. Trying to explicitly relate the holomorphic curve $[\hat{g}_A]$ to its image quadric does in general however not seem to give manageable expressions.

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