GL(2)-structures in dimension four, *H*-flatness and integrability

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Abstract. We show that torsion-free four-dimensional GL(2)-structures are flat up to a coframe transformation with a mapping taking values in a certain subgroup $H \subset \mathrm{SL}(4,\mathbb{R})$, which is isomorphic to a semidirect product of the three-dimensional continuous Heisenberg group $H_3(\mathbb{R})$ and the Abelian group $\mathbb{R}$. In addition, we show that the relevant PDE system is integrable in the sense that it admits a dispersionless Lax-pair.

1. Introduction

A GL(2)-structure on a smooth 4-manifold M is given by a smoothly varying family of twisted cubic curves, one in each projectivised tangent space of M. Equivalently, a GL(2)-structure is the same as G-structure $\pi\colon B \to M$ on M, where G is the image subgroup of the faithful irreducible 4-dimensional representation of $\mathrm{GL}(2,\mathbb{R})$ on the space of homogeneous polynomials of degree three with real coefficients in two real variables. A GL(2)-structure is called *torsion-free* if its associated G-structure is torsion-free. Torsion-free GL(2)-structures are of particular interest, as they provide examples of torsion-free connections with exotic holonomy group $\mathrm{GL}(2,\mathbb{R})$. However, the local existence of torsion-free GL(2)-structures is highly non-trivial, even when applying the Cartan–Kähler machinery, which is particularly well-suited for the construction of torsion-free connections with special holonomy. Adapting methods of Hitchin [10], Bryant [2] gave an elegant twistorial construction of real-analytic torsion-free GL(2)-structures in dimension four, thus providing the first example of an irreducibly-acting holonomy group of a (non-metric) torsion-free connection missing from Berger's list [1] of such connections.

A natural source for GL(2)-structures are differential operators. Recall that the principal symbol $\sigma(\mathrm{D})$ of a k-th order linear differential operator $\mathrm{D}\colon C^\infty(M,\mathbb{R}^n) \to C^\infty(M,\mathbb{R}^m)$ assigns to each point $p \in M$ a homogeneous polynomial of degree k on T^*_pM, with values in $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$. Therefore, in each projectivised cotangent space $\mathbb{P}(T^*_pM)$ of M we obtain the so-called *characteristic variety* Ξ_p of D, consisting of those $[\xi] \in \mathbb{P}(T^*_pM)$, for which the linear mapping $\sigma_\xi(\mathrm{D})\colon \mathbb{R}^n \to \mathbb{R}^m$ fails to be injective. Given a (possibly non-linear) differential operator D and a smooth $\mathbb{R}^n$-valued function u defined on some open subset $U \subset M$ and which satisfies $\mathrm{D}(u) = 0$, we may ask that the linearisation $\mathrm{L}_u(\mathrm{D})$ of D around u has characteristic varieties all of which are

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the tangential variety of the twisted cubic curve. Consequently, one obtains a GL(2)-structure on the domain of definition of each solution u of the PDE $\mathrm{D}(u) = 0$ for an appropriate class of differential operators. Various examples of such operators have recently been given by Ferapontov–Kruglikov [7]. In particular, they show that locally all torsion-free GL(2)-structures arise in this fashion for some *second order* operator D, which furthermore has the property that the PDE $\mathrm{D}(u) = 0$ admits a dispersionless Lax representation. We also refer the reader to [8] for an application of similar ideas to the case of three-dimensional Einstein–Weyl structures.

Here we show that if a 4-manifold M carries a torsion-free GL(2)-structure $\pi\colon B \to M$, then for every point $p \in M$ there exists a p-neighbourhood U_p, local coordinates $x\colon U_p \to \mathbb{R}^4$ and a mapping $h\colon U_p \to H$ into a certain 4-dimensional subgroup $H \subset \mathrm{SL}(4,\mathbb{R})$, so that the coframing $\eta = h\,\mathrm{d}x$ is a local section of $\pi\colon B \to M$. The group H is isomorphic to a semidirect product of the three-dimensional continuous Heisenberg group $H_3(\mathbb{R})$ and the Abelian group $\mathbb{R}$. Moreover, the mapping h satisfies a *first order* quasi-linear PDE system which admits a dispersionless Lax-pair. As in [7], linearising the PDE system around a solution h gives a linear first order differential operator whose characteristic variety is the tangential variety of the twisted cubic curve. Also, note that our result shows that 4-dimensional torsion-free GL(2)-structures are *H-flat*, that is, flat up to a coframe transformation with a mapping taking values in H.

Along the way (see Theorem 2.4), we derive a first order PDE describing general H-flat torsion-free G-structures which may be of independent interest.

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2. G-structures and H-flatness

In this section we collect some elementary facts about G-structures, introduce the notion of H-flatness and derive the first order PDE system describing H-flat torsion-free G-structures. Throughout the article all manifolds and maps are assumed to be smooth, that is C^∞.

2.1. The coframe bundle and G-structures

Let M be an n-manifold and V a real n-dimensional vector space. A V-valued coframe at $p \in M$ is a linear isomorphism $f\colon T_pM \to V$. The set F_pM of V-valued coframes at $p \in M$ is the fibre of the principal right $\mathrm{GL}(V)$ coframe bundle $\upsilon\colon FM \to M$, where the right action $R_a\colon FM \to FM$ is defined by the rule $R_a(f) = a^{-1} \circ f$ for all $a \in \mathrm{GL}(V)$ and $f \in FM$. Of course, we may identify $V \simeq \mathbb{R}^n$, but it is often advantageous to allow V to be an abstract vector space, in which case we say FM is *modelled* on V. The coframe bundle carries a tautological V-valued 1-form defined by $\omega_f = f \circ \upsilon_*$ so that we have the equivariance property $R_a^*\omega = a^{-1}\omega$. A local υ-section $\eta\colon U \to FM$ is called a *coframing* on $U \subset M$ and a choice of a basis of V identifies η with n linearly independent 1-forms on U.

Let $G \subset \mathrm{GL}(V)$ be a closed subgroup. A G-structure on M is a reduction $\pi\colon B \to M$ of the coframe bundle with structure group G, equivalently, a smooth section of the fibre bundle $FM/G \to M$. For local considerations we may take $M = V$. Note that in this case M is equipped with a coframing η_0 defined by the exterior derivative of the identity map $\eta_0 = \mathrm{d}\,\mathrm{Id}_V$. Consequently, the coframe bundle of V may naturally be identified with $V \times \mathrm{GL}(V)$ and hence the set of G-structures on V is in one-to-one correspondence with the space of smooth maps $V \to \mathrm{GL}(V)/G$. In particular, a smooth map $h\colon V \to \mathrm{GL}(V)$ defines a G-structure on V by composing h with the quotient projection $\mathrm{GL}(V) \to \mathrm{GL}(V)/G$.

2.2. H-flatness

A G-structure $\pi\colon B \to M$ is called *flat* if in a neighbourhood U_p of every point $p \in M$ there exist local coordinates $x\colon U_p \to V$, so that $\mathrm{d}x\colon U_p \to FM$ takes values in B. We remark that flat G-structures also are often called *integrable.* Suppose $H \subset \mathrm{GL}(V)$ is a closed subgroup. We say a G-structure is *H-flat* if in a neighbourhood U_p of every point $p \in M$ there exist local coordinates $x\colon U_p \to V$ and a mapping $h\colon U_p \to H$, so that $h\,\mathrm{d}x\colon U_p \to FM$ takes values in B. Clearly, every G-structure is $\mathrm{GL}(V)$-flat and a G-structure is flat in the usual sense if and only if it is $\{e\}$-flat, where $\{e\}$ denotes the trivial subgroup of $\mathrm{GL}(V)$.

Example 2.1. Every $\mathrm{O}(2)$-structure is $\mathbb{R}^+$-flat, where $\mathbb{R}^+$ denotes the group of uniform scaling transformations of $\mathbb{R}^2$ with positive scale factor. This is the existence of local isothermal coordinates for Riemannian metrics in two-dimensions. Likewise, conformally flat Riemannian metrics in dimensions $n > 2$ yield examples of $\mathrm{O}(n)$-structures that are $\mathbb{R}^+$-flat.

Remark 2.2. Note that if a G-structure is H-flat for some Lie group $H \subset G$, then it is $\{e\}$-flat.

2.3. A PDE for H-flat torsion-free G-structures

A G-structure $\pi\colon B \to M$ is called *torsion-free* if there exists a principal G-connection θ on B, so that Cartan's first structure equation

$$\mathrm{d}\omega = -\theta \wedge \omega \tag{1}$$

holds. Recall that a principal G-connection on B is a 1-form θ on B with values in the Lie algebra $\mathfrak{g}$ of G that pulls back to each π-fibre to be the canonical left invariant 1-form on G and that is equivariant with respect to the adjoint action of G, that is, θ satsifies $R_g^*\theta = \mathrm{Ad}(g^{-1})\theta$ for all $g \in G$.

Remark 2.3. We remark that a weaker notion of torsion-freeness is also in use, see for instance [3, 11]. Namely, a G-structure $\pi\colon B \to M$ is called torsion-free if there exists a $\mathfrak{g}$-valued 1-form θ on B so that (1) holds.

We may ask when a G-structure on V induced by a mapping $h\colon V \to H \subset \mathrm{GL}(V)$ is torsion-free. To this end let $A \subset V^* \otimes V$ be a linear subspace. Denote by

$$\delta\colon V^* \otimes V^* \otimes V \to \Lambda^2(V^*) \otimes V$$

the natural skew-symmetrisation map. Recall that the Spencer cohomology group $H^{0,2}(A)$ of A is the quotient

$$H^{0,2}(A) = \left(\Lambda^2(V^*) \otimes V\right)/\delta(V^* \otimes A).$$

Let

$$\Pi_A\colon \Lambda^2(V^*) \otimes V \to H^{0,2}(A)$$

denote the quotient projection and let μ_H denote the Maurer–Cartan form of H. Note that $\psi_h = h^*\mu_H$ is a 1-form on V with values in the Lie algebra $\mathfrak{h}$ of H, that is, a smooth map

$$\psi_h\colon V \to V^* \otimes \mathfrak{h} \subset V^* \otimes \mathfrak{gl}(V) \simeq V^* \otimes V^* \otimes V.$$

We define $\tau_h = \delta\,\psi_h$, so that τ_h is a 2-form on V with values in V. We now have:

Theorem 2.4. *Let $h\colon V \to H$ be a smooth map. Then the G-structure defined by h is torsion-free if and only if*

$$\Pi_{\mathrm{Ad}(h^{-1})\mathfrak{g}}\,\tau_h = 0. \tag{2}$$

Remark 2.5. In the case where $H = G$ the H-structure defined by h is the same as the torsion-free H-structure defined by the map $h \equiv \mathrm{Id}_V\colon V \to \mathrm{GL}(V)$, hence (2) must be trivially satisfied. This is indeed the case. Since the adjoint action of H preserves $\mathfrak{h}$, we obtain for any map $h\colon V \to H$

$$\Pi_{\mathrm{Ad}(h^{-1})\mathfrak{h}}\,\tau_h = \Pi_{\mathfrak{h}}\,\tau_h = \Pi_{\mathfrak{h}}\,\delta\,\psi_h = 0.$$

Proof of Theorem 2.4. For the proof we fix an identification $V \simeq \mathbb{R}^n$. Let $x = (x^i)$ denote the standard linear coordinates on $\mathbb{R}^n$. Furthermore let $h\colon \mathbb{R}^n \to H \subset \mathrm{GL}(n,\mathbb{R})$ be given and let $\pi\colon B_h \to \mathbb{R}^n$ denote the G-structure defined by h, that is,

$$B_h = \left\{(x,a) \in \mathbb{R}^n \times \mathrm{GL}(n,\mathbb{R}) \,:\, a = h^{-1}(x)g,\ g \in G\right\}.$$

We have a G-bundle isomorphism

$$\psi\colon \mathbb{R}^n \times G \to B_h, \quad (x,g) \mapsto (x, h^{-1}(x)g).$$

The tautological 1-form ω_0 on $F\mathbb{R}^n \simeq \mathbb{R}^n \times \mathrm{GL}(n,\mathbb{R})$ satisfies $(\omega_0)_{(x,a)} = a^{-1}\mathrm{d}x$ for all $(x,a) \in \mathbb{R}^n \times \mathrm{GL}(n,\mathbb{R})$. Continuing to write ω_0 for the pullback to B_h of ω_0, we obtain

$$\omega_{(x,g)} := (\psi^*\omega_0)_{(x,g)} = g^{-1}h(x)\mathrm{d}x.$$

Let α be any 1-form on $\mathbb{R}^n$ with values in $\mathfrak{g}$, the Lie-algebra of G. We obtain a principal G-connection $\theta = (\theta^i_j)$ on $\mathbb{R}^n \times G$ by defining

$$\theta = g^{-1}\alpha g + g^{-1}\mathrm{d}g,$$

where $g\colon \mathbb{R}^n \times G \to G \subset \mathrm{GL}(n,\mathbb{R})$ denotes the projection onto the latter factor. Conversely, every principal G-connection on the trivial G-bundle

$\mathbb{R}^n \times G$ arises in this fashion. The G-structure B_h is torsion-free if and only if there exists a principal G-connection θ such that

$$\mathrm{d}\omega + \theta \wedge \omega = 0,$$

which is equivalent to

$$0 = \mathrm{d}\left(g^{-1}h\mathrm{d}x\right) + \left(g^{-1}\alpha g + g^{-1}\mathrm{d}g\right) \wedge g^{-1}h\mathrm{d}x$$

or

$$0 = \left(\mathrm{d}g^{-1} + g^{-1}\mathrm{d}gg^{-1}\right) \wedge h\,\mathrm{d}x + g^{-1}\left(\mathrm{d}h \wedge \mathrm{d}x + \alpha \wedge h\,\mathrm{d}x\right).$$

Using $0 = \mathrm{d}\left(g^{-1}g\right)$, we see that the G-structure defined by h is torsion-free if and only if there exists a 1-form α on V with values in $\mathfrak{g}$ such that

$$0 = \mathrm{d}h \wedge \mathrm{d}x + \alpha \wedge h\,\mathrm{d}x.$$

This is equivalent to

$$\left(h^{-1}\mathrm{d}h + h^{-1}\alpha h\right) \wedge \mathrm{d}x = 0$$

or

$$\left(\psi_h + \mathrm{Ad}(h^{-1})\alpha\right) \wedge \mathrm{d}x = 0, \tag{3}$$

where $\psi_h = h^{-1}\mathrm{d}h$ denotes the h-pullback of the Maurer–Cartan form of H and $\mathrm{Ad}(h)v = hvh^{-1}$ the adjoint action of $h \in H$ on $v \in \mathfrak{gl}(n, \mathbb{R})$. Now (3) is equivalent to

$$\delta\,\psi_h + \delta\,\mathrm{Ad}(h^{-1})\alpha = 0.$$

Since α takes values in $\mathfrak{g}$, this implies that $\tau_h = \delta\,\psi_h$ lies in the δ-image of $V^* \otimes \mathrm{Ad}(h^{-1})\mathfrak{g}$. Therefore, we obtain

$$\Pi_{\mathrm{Ad}(h^{-1})\mathfrak{g}}\,\tau_h = 0.$$

Conversely, suppose τ_h lies in the δ-image of $V^* \otimes \mathrm{Ad}(h^{-1})\mathfrak{g}$. Then there exists a 1-form β on V with values in $h^{-1}\mathfrak{g}h$ so that

$$\tau_h = \delta\,\psi_h = \delta\,\beta.$$

Hence, the $\mathfrak{g}$-valued 1-form α on V defined by $\alpha = -h\beta h^{-1}$ satisfies

$$\tau_h + \delta\,h^{-1}\alpha h = \delta\,\psi_h + \delta\,\mathrm{Ad}(h^{-1})\alpha = 0,$$

thus proving the claim. □

3. GL(2)-structures

Let x, y denote the standard linear coordinates on $\mathbb{R}^2$ and let $\mathbb{R}[x, y]$ denote the polynomial ring with real coefficients generated by x and y. We let $\mathrm{GL}(2, \mathbb{R})$ act from the left on $\mathbb{R}[x, y]$ via the usual linear action on x, y. We denote by $\mathcal{V}_d$ the subspace consisting of homogeneous polynomials in degree $d \geqslant 0$ and by $G_d \subset \mathrm{GL}(\mathcal{V}_d)$ the image subgroup of the $\mathrm{GL}(2, \mathbb{R})$ action on $\mathcal{V}_3$. The vector space $\mathcal{V}_3$ carries a two-dimensional cone $\tilde{\mathcal{C}}$ of distinguished polynomials, consisting of the perfect cubes, i.e., those that are of the form $(ax + by)^3$ for $ax + by \in \mathcal{V}_1$. The reader may easily check that G_3 is characterised as the subgroup of $\mathrm{GL}(\mathcal{V}_3)$ that preserves $\tilde{\mathcal{C}}$. The projectivisation of $\tilde{\mathcal{C}}$ gives an algebraic curve $\mathcal{C}$ of degree 3 in $\mathbb{P}(\mathcal{V}_3)$, which is linearly equivalent to the *twisted cubic curve*, i.e., the curve in $\mathbb{RP}^3$ defined by the zero locus of the three homogeneous polynomials

$$P_0 = XZ - Y^2, \quad P_1 = YW - Z^2, \quad P_2 = XW - YZ,$$

where $[X : Y : Z : W]$ are the standard homogeneous coordinates on $\mathbb{RP}^3$. The vector space $\mathcal{V}_3$ carries another algebraic variety in its projectivisation besides the twisted cubic curve. Indeed, the polynomials having vanishing discriminant define a G_3-invariant quartic cone $\tilde{\mathcal{Q}}$ whose projectivisation $\mathcal{Q}$ defines a quartic hypersurface in $\mathbb{P}(\mathcal{V}_3)$. Furthermore, the singular locus of $\mathcal{Q}$ is the twisted cubic curve $\mathcal{C}$ and the tangential variety of $\mathcal{C}$ is $\mathcal{Q}$.

Let M be a 4-manifold and let $\upsilon\colon FM \to M$ denote its coframe bundle modelled on $\mathcal{V}_3$. A GL(2)-structure on M is a reduction $\pi\colon B \to M$ of FM with structure group $G_3 \simeq \mathrm{GL}(2,\mathbb{R})$. By definition, a GL(2)-structure identifies each tangent space of M with $\mathcal{V}_3$ up to the action by $\mathrm{GL}(2,\mathbb{R})$. Consequently, each projectivised tangent space $\mathbb{P}(T_pM)$ of M carries an algebraic curve $\mathcal{C}_p$, which is linearly equivalent to the twisted cubic curve. Conversely, if $\mathcal{C} \subset \mathbb{P}(TM)$ is a smooth subbundle having the property that each fibre $\mathcal{C}_p$ is linearly equivalent to the twisted cubic curve, then one obtains a unique reduction of the coframe bundle of M whose structure group is G_3.

For what follows it will be convenient to identify $\mathcal{V}_3 \simeq \mathbb{R}^4$ by the isomorphism $\mathcal{V}_3 \to \mathbb{R}^4$ defined on the basis of monomials as

$$x^{(3-i)}y^i \mapsto e_{i+1},$$

where $i = 0, 1, 2, 3$ and e_i denotes the standard basis of $\mathbb{R}^4$. Note that, under the identification $T_pM = \mathcal{V}_3$, the cone $\tilde{\mathcal{C}}$ of a GL(2)-structure at p can be written as

$$\tilde{\mathcal{C}}_p = \{s^3e_1 + 3s^2te_2 + 3st^2e_3 + t^3e_4 \mid s, t \in \mathbb{R}\}.$$

We now have:

Theorem 3.1. *All torsion-free* GL(2)*-structures in dimension four are H-flat, where $H \subset \mathrm{SL}(4,\mathbb{R})$ is the subgroup consisting of matrices of the form*

$$\begin{pmatrix} 1 & A & B & D \\ 0 & 1 & A & C \\ 0 & 0 & 1 & A \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

and where A, B, C, D are arbitrary real numbers.

Remark 3.2. We note that the group H is isomorphic to a semidirect product of the continuous three-dimensional Heisenberg group $H_3(\mathbb{R})$ and the Abelian group $\mathbb{R}$, that is, $H \simeq H_3(\mathbb{R}) \rtimes \mathbb{R}$. Indeed, $H_3(\mathbb{R})$ has a faithful (necessarily reducible) four-dimensional representation defined by the Lie group homomorphism $\varphi\colon H_3(\mathbb{R}) \to \mathrm{SL}(4,\mathbb{R})$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & \frac{1}{2}a^2 + b & \frac{1}{6}a^3 + ab - c \\ 0 & 1 & a & \frac{1}{2}a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The homomorphism φ embeds $H_3(\mathbb{R})$ as a normal subgroup of the group H and we think of $\mathbb{R}$ as the Abelian subgroup of H defined by setting $A = B = D = 0$ in (4).

Remark 3.3. In fact, the notion of a GL(2)-structure makes sense in all dimensions $d \geqslant 3$. However, torsion-free GL(2)-structures in dimensions exceeding four are $\{e\}$-flat [**2**], that is, flat in the usual sense. We refer the reader to [**9, 18**] for a comprehensive study of five-dimensional GL(2)-structures (with torsion).

Remark 3.4. Phrased differently, Theorem 3.1 states that locally every torsion-free GL(2)-structure in dimension four is obtained from a solution to the first order PDE system (2), where h takes values in the aforementioned group H.

Proof of Theorem 3.1. We shall prove that for a given torsion-free GL(2)-structure one can always choose local coordinates such that the cone $\tilde{\mathcal{C}}$ has the following form

$$\tilde{\mathcal{C}} = \{\ s^3V_0 + 3s^2tV_1 + 3st^2V_2 + t^3V_3|\ s,t \in \mathbb{R}\},$$

where the framing (V_0, V_1, V_2, V_3) is

$$\begin{aligned} &V_0 = \partial_0, \qquad V_1 = \partial_1 + \alpha\partial_0, \qquad V_2 = \partial_2 + \alpha\partial_1 + \beta\partial_0,\\ &V_3 = \partial_3 + \alpha\partial_2 + \gamma\partial_1 + \delta\partial_0, \end{aligned} \tag{5}$$

for some functions α, β, γ and δ. Then, the dual coframing is of the form $h\,dx$, where h takes values in H with

$$A = -\alpha, \quad B = -\beta + \alpha^2, \quad C = -\gamma + \alpha^2, \quad D = -\delta + \alpha(\gamma + \beta) - \alpha^3.$$

In order to derive the desired form of $\tilde{\mathcal{C}}$ we explore a correspondence between the torsion-free GL(2)-structures and classes of contact equivalent fourth order ODEs (compare the proof of [**4, Theorem 1**] and a similar correspondence in dimension 3). Indeed, it is proved in [**2**] that any torsion-free GL(2)-structure is defined by a fourth order ODE of the form

$$x^{(4)} = F(y, x, x', x'', x'''), \tag{6}$$

where the function $F = F(y, x_0, x_1, x_2, x_3)$ satisfies a system of non-linear equations that we will refer to as the Bryant–Wünschmann condition. (Similar conditions in higher dimensions are known as the generalized Wünschmann conditions, because they generalize the classical 3-dimensional case, c.f. [**6, 17**].)

Above, (y, x_0, x_1, x_2, x_3) denote the standard coordinates on the space $J^3(\mathbb{R},\mathbb{R})$ of 3-jets of functions $\mathbb{R} \to \mathbb{R}$ and the Bryant–Wünschmann condition is invariant with respect to the group of contact transformations of the coordinates. The GL(2)-structure corresponding to equation (6) is defined on the solution space of (6), i.e., on the quotient space $J^3(\mathbb{R},\mathbb{R})/X_F$, where $X_F = \partial_y + x_1\partial_0 + x_2\partial_1 + x_3\partial_2 + F\partial_3$ is the total derivative. In order to define the structure, we first consider the following field of cones on $J^3(\mathbb{R},\mathbb{R})$ as in [**12**]

$$\hat{\mathcal{C}} = \{\ s^3\hat{V}_0 + 3s^2t\hat{V}_1 + 3st^2\hat{V}_2 + t^3\hat{V}_3 \mid s,t \in \mathbb{R}\} \quad \mathrm{mod}\ X_F$$

where

$$\hat{V}_0 = \frac{3}{4}\partial_3,$$
$$\hat{V}_1 = \frac{1}{2}\partial_2 + \frac{3}{8}\partial_3 F\partial_3,$$
$$\hat{V}_2 = \frac{1}{2}\partial_1 + \frac{1}{4}\partial_3 F\partial_2 + \left(\frac{7}{20}\partial_2 F - \frac{3}{20}X_F(\partial_3 F) + \frac{9}{40}(\partial_3 F)^2\right)\partial_3,$$
$$\hat{V}_3 = \partial_0 + \frac{1}{4}\partial_3 F\partial_1 + \left(\partial_2 F - \frac{5}{4}X_F(\partial_3 F) + \frac{7}{16}(\partial_3 F)^2 + \frac{7}{10}K\right)\partial_2$$
$$+ \Big(\partial_1 F - \frac{3}{10}X_F(K) - X_F(\partial_2 F) + \frac{21}{40}K\partial_3 F$$
$$- \frac{27}{16}X_F(\partial_3 F)\partial_3 F - \frac{3}{4}\partial_2 F\partial_3 F + \frac{3}{4}X_F^2(\partial_3 F) + \frac{27}{64}(\partial_3 F)^3\Big)\partial_3,$$

with $K = -\partial_2 F + \frac{3}{2}X(\partial_3 F) - \frac{3}{8}(\partial_3 F)^2$. To define the cone one looks for (f, g) such that

$$\mathrm{ad}^4_{fX_F}(g\partial_3) = 0 \mod X_F, \partial_3, \partial_2, \tag{7}$$

where $\mathrm{ad}^i_{X_F}$ stands for the iterated Lie bracket with the vector field X_F. Then $\hat{\mathcal{C}}_p$ is defined as the set of all $(\mathrm{ad}^3_{fX_F}(g\partial_3))(p)$, where (f, g) solve (7). The explicit formula for $\hat{\mathcal{C}}$ can be found using [**12, Proposition 4.1**] and [**12, Corollary 5.3**]. The cone $\hat{\mathcal{C}}$ is invariant with respect to the flow of X_F if and only if (6) satisfies the Bryant–Wünschmann condition. In this case (7) takes the form $\mathrm{ad}^4_{fX_F}(g\partial_3) = 0 \mod X_F$ (c.f. [**13**]). Then $\hat{\mathcal{C}}$ can be projected to the quotient space $J^3(\mathbb{R}, \mathbb{R})/X_F$ and defines a GL(2)-structure there via the field of cones $\tilde{\mathcal{C}} = q_*\hat{\mathcal{C}}$, where $q\colon J^3(\mathbb{R}, \mathbb{R}) \to J^3(\mathbb{R}, \mathbb{R})/X_F$ is the quotient map. Note that $J^3(\mathbb{R}, \mathbb{R})/X_F$ can be identified with the hypersurface $\{y = 0\} \subset J^3(\mathbb{R}, \mathbb{R})$. Denoting

$$\alpha = \partial_3 F|_{y=0},$$
$$\beta = \left(\frac{7}{20}\partial_2 F - \frac{3}{20}X(\partial_3 F) + \frac{9}{40}(\partial_3 F)^2\right)\Big|_{y=0},$$
$$\gamma = \left(\partial_2 F - \frac{5}{4}X_F(\partial_3 F) + \frac{7}{16}(\partial_3 F)^2 + \frac{7}{10}K\right)\Big|_{y=0},$$
$$\delta = \Big(\partial_1 F - \frac{3}{10}X(K) - X(\partial_2 F) + \frac{21}{40}K\partial_3 F - \frac{27}{16}X(\partial_3 F)\partial_3 F$$
$$-\frac{3}{4}\partial_2 F\partial_3 F + \frac{3}{4}X^2(\partial_3 F) + \frac{27}{64}(\partial_3 F)^3\Big)\Big|_{y=0}$$

we get that

$$\tilde{\mathcal{C}} = \{\, s^3 V_0 + 3s^2 t V_1 + 3st^2 V_2 + t^3 V_3 \mid s, t \in \mathbb{R}\}$$

where

$$V_0 = \frac{3}{4}\partial_3, \qquad V_1 = \frac{1}{2}\partial_2 + \frac{3}{8}\alpha\partial_3, \qquad V_2 = \frac{1}{2}\partial_1 + \frac{1}{4}\alpha\partial_2 + \beta\partial_3,$$
$$V_3 = \partial_0 + \frac{1}{4}\alpha\partial_1 + \gamma\partial_2 + \delta\partial_3.$$

The following linear change of coordinates

$$(x_0, x_1, x_2, x_3) \mapsto \left(x_3, 2x_2, 2x_1, \frac{4}{3}x_0\right)$$

transforms (V_0, V_1, V_2, V_3) to

$$V_0 = \partial_0, \qquad V_1 = \partial_1 + \frac{1}{2}\alpha\partial_0, \qquad V_2 = \partial_2 + \frac{1}{2}\alpha\partial_1 + \frac{4}{3}\beta\partial_0,$$
$$V_3 = \partial_3 + \frac{1}{2}\alpha\partial_2 + 2\gamma\partial_1 + \frac{4}{3}\delta\partial_0,$$

which is equivalent to (5) up to constants. □

Remark 3.5. Theorem 3.1 should be compared with [**7, Proposition 1**], which can be rephrased that locally any torsion-free GL(2)-structure admits a coframing of the form $h\,\mathrm{d}x$ with

$$h = \begin{pmatrix} a_1a_2a_3 & a_0a_2a_3 & a_0a_1a_3 & a_0a_1a_2 \\ \begin{matrix}\frac{1}{3}(a_1a_2b_3 + a_1b_2a_3 \\ +b_1a_2a_3)\end{matrix} & \begin{matrix}\frac{1}{3}(a_0a_2b_3 + a_0b_2a_3 \\ +b_0a_2a_3)\end{matrix} & \begin{matrix}\frac{1}{3}(a_0a_1b_3 + a_0b_1a_2 \\ +b_0a_1a_3)\end{matrix} & \begin{matrix}\frac{1}{3}(a_0a_1b_2 + a_0b_1a_3 \\ +b_0a_1a_2)\end{matrix} \\ \begin{matrix}\frac{1}{3}(a_1b_2b_3 + b_1a_2b_3 \\ +b_1b_2a_3)\end{matrix} & \begin{matrix}\frac{1}{3}(a_0b_2b_3 + b_0a_2b_3 \\ +b_0b_2a_3)\end{matrix} & \begin{matrix}\frac{1}{3}(a_0b_1b_3 + b_0a_1b_3 \\ +b_0b_1a_3)\end{matrix} & \begin{matrix}\frac{1}{3}(a_0b_1b_2 + b_0a_1b_2 \\ +b_0a_1b_2)\end{matrix} \\ b_1b_2b_3 & b_0b_2b_3 & b_0b_1b_3 & b_0b_1b_2 \end{pmatrix},$$

where $a_i = \left(\frac{\partial u}{\partial x_i}\right)^{-1}$ and $b_i = \left(\frac{\partial v}{\partial x_i}\right)^{-1}$ for some real-valued functions u and v on $\mathcal{V}_3 \simeq \mathbb{R}^4$. One checks that h is not contained in any proper subgroup of $\mathrm{GL}(4,\mathbb{R})$. It is an interesting problem to find the smallest possible dimension of the group H, such that all torsion-free GL(2)-structures are H-flat (we believe that dimension 4 from Theorem 3.1 is optimal).

4. Integrability

In this section we derive the system (2) explicitly in terms of the functions A, B, C and D of Theorem 3.1. Moreover, we prove that it possesses a dispersionless Lax pair understood as a pair of commuting vector fields depending on a spectral parameter. Systems of this type, e.g., the dispersionless Kadomtsev-Petviashivili equation, often appear as dispersionless limits of integrable PDEs. Other examples include the Plebański heavenly equation or the Manakov-Santini system describing 3-dimensional Einstein-Weyl geometry. We refer to [**15, 16**] for general methods of integration of such systems. Let $H \subset \mathrm{SL}(4,\mathbb{R})$ be the subgroup of matrices (4). Furthermore, let A_i, B_i, C_i and D_i denote $\partial_i A$, $\partial_i B$, $\partial_i C$ and $\partial_i D$, respectively,

Theorem 4.1. *An H-flat* $\mathrm{GL}(2)$*-structure defined by a coframing $h\,\mathrm{d}x$, where h takes values in H, is torsion-free if and only if*

$$
\begin{aligned}
&V_2(D) - V_3(B) - AV_2(B) - CV_2(A) + AV_3(A) + A^2V_2(A) = 0\\
&2V_1(D) - V_2(C) - 2AV_1(B) - V_3(A)+\\
&\qquad\qquad + AV_2(A) + 2A^2V_1(A) - 2CV_1(A) = 0\\
&V_0(D) - 2V_1(C) + 3V_1(B) - AV_0(B) - 2V_2(A)\\
&\qquad\qquad - AV_1(A) - CV_0(A) + A^2V_0(A) = 0\\
&V_0(C) - 2V_0(B) + V_1(A) + AV_0(A) = 0,
\end{aligned}
\tag{8}
$$

and where the framing (V_0, V_1, V_2, V_3) dual to $h\,\mathrm{d}x$ is explicitly given by

$$
\begin{aligned}
&V_0 = \partial_0, \qquad V_1 = \partial_1 - A\partial_0, \qquad V_2 = \partial_2 - A\partial_1 - (B - A^2)\partial_0,\\
&V_3 = \partial_3 - A\partial_2 - (C - A^2)\partial_1 - (D - (C + B)A + A^3)\partial_0.
\end{aligned}
$$

The system (8) can be put in the Lax form $[L_0, L_1] = 0$ with

$$
\begin{aligned}
L_0 &= \partial_3 + (-C + 2A\lambda - 3\lambda^2)\partial_1\\
&\quad + (-D + AC - 2A^2\lambda + 4A\lambda^2 - 2\lambda^3)\partial_0 + \nu(\lambda)\partial_\lambda,\\
L_1 &= \partial_2 + (-A + 2\lambda)\partial_1 + (-B + A^2 - 2A\lambda + \lambda^2)\partial_0 + \mu(\lambda)\partial_\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
\nu(\lambda) = &\Big(\frac{1}{2}A^2A_1 - ABA_0 + AA_2 - AB_1 - \frac{1}{2}DA_0 - \frac{1}{2}C_2\\
&\qquad + \frac{1}{2}AC_1 + \frac{1}{2}BC_0 - \frac{1}{2}CA_1 + \frac{1}{2}ACA_0 + \frac{1}{2}A_3\Big)\\
&+ (3B_1 - C_1 - AA_1 - AC_0 + 2BA_0 - 2A_2)\lambda\\
&+ (C_0 - A_1)\lambda^2\\
\mu(\lambda) = &\Big(\frac{1}{2}AA_1 + \frac{1}{2}AC_0 - BA_0 + A_2 - B_1\Big)\\
&+ \Big(\frac{1}{2}A_1 - \frac{1}{2}C_0\Big)\lambda,
\end{aligned}
$$

for some auxiliary spectral coordinate λ.

Remark 4.2. The spectral parameter λ can be treated as an affine parameter on the fibres of $\mathcal{C}$. The theorem states that $\mathcal{D} = \mathrm{span}\{L_0, L_1\}$ is an integrable rank-2 distribution on $\mathcal{C}$. There is a 3-parameter family of integral manifolds of $\mathcal{D}$. Projections of these submanifolds to M give a 3-parameter family of 2-dimensional submanifolds of M tangent to the field of cones $\tilde{\mathcal{C}}$.

Remark 4.3. The space of integral manifolds of the aforementioned distribution $\mathcal{D} = \mathrm{span}\{L_0, L_1\}$ is the twistor space T of a torsion-free GL(2)-structure. In this context $\mathcal{C}$ is the correspondence space and we have a double fibration picture $M \longleftarrow \mathcal{C} \longrightarrow T$, where the fibres of the second projection are tangent to $\mathcal{D}$. If the coefficients μ and ν in the Lax pair (L_0, L_1) vanish, then there is additional natural projection, defined by the parameter λ, from T to one-dimensional projective space. In other words, for any fixed λ, the integral leaves of $\mathcal{D}_\lambda = \mathrm{span}\{L_0(\lambda), L_1(\lambda)\}$ define a 2-dimensional foliation of M. Among these structures there is a subclass for which the distribution $\mathrm{span}\{L_0(\lambda), L_1(\lambda), \frac{d}{d\lambda}L_1(\lambda)\}$ is integrable and thus defines a 3-dimensional foliation. Such foliations are known as Veronese webs, c.f. [**13**]. From this point of view, the Veronese webs can be thought of as higher-dimensional counterparts of 3-dimensional hyper-CR Einstein-Weyl structures [**5**].

Veronese webs are described by a hierarchy of integrable systems introduced in [**5**], which generalize the dispersionless Hirota equation. It is worth seeing how the system (8) looks like in this case. For this we note that the H-flat form of 4-dimensional Veronese webs has been given in [**14, Section 6**] and in this case we get (after permutation of indices) the following coefficients

$$A = \frac{\partial_1 f}{\partial_0 f}, \qquad B = C = \frac{\partial_2 f}{\partial_0 f}, \qquad D = \frac{\partial_3 f}{\partial_0 f},$$

where $f = f(x_0, x_1, x_2, x_3)$ is a function. Then, in terms of f, the system (8) takes the following simple form

$$\begin{aligned} f_2 f_{00} - f_0 f_{02} - f_1 f_{01} + f_0 f_{11} &= 0, \\ f_3 f_{00} - f_0 f_{03} - f_1 f_{02} + f_0 f_{12} &= 0, \\ f_3 f_{01} - f_0 f_{13} - f_2 f_{02} + f_0 f_{22} &= 0, \end{aligned}$$

which coincides with the system derived in [**14, Theorem 6.1**]. One can also set $H_i = -\frac{f_{i+1}}{f_0}$ and pass to a system derived in [**14, Theorem 6.2**]. An example of such a structure is given by the equation $x^{(4)} = (x^{(3)})^{4/3}$ from [**5**]. In this case, using the formulae given in the proof of Theorem 3.1, one finds $\alpha = x_0^{1/3}$, $\beta = \gamma = x_0^{2/3}$ and $\delta = x_0$. Thus $A = -x_0^{1/3}$, $B = C = D = 0$ and $f(x_0, x_1, x_2, x_3) = x_1 - \frac{3}{2}x_0^{2/3}$.

Remark 4.4. A Cartan–Kähler analysis reveals that the first order system (8) – or equivalently (2) – is involutive and has solutions depending on four functions of three variables, confirming the count of Bryant [**2**]. Moreover, straightforward computations show that the characteristic variety of the system (8) linearised along any solution (A, B, C, D) is the discriminant locus $\mathcal{Q}$, i.e., the tangential variety of $\mathcal{C}$.

Proof of Theorem 4.1. The system (8) can be directly obtained by expanding (2) explicitly in terms of the functions A, B, C, D. Here we use a different method and apply [**12, Corollary 7.4**] to the framing $(V_0, 3V_1, 3V_2, V_3)$. Namely, denoting $\lambda = \frac{s}{t}$, we get that the curve $\mathcal{C}$ in $\mathbb{P}(TM)$ is the image of $\lambda \mapsto \mathbb{R}V(\lambda) \in \mathbb{P}(TM)$, where $V(\lambda) = \lambda^3 V_0 + 3\lambda^2 V_1 + 3\lambda V_2 + V_3$ and the vector fields V_0, V_1, V_2 and V_3 are given by (5) with

$$\alpha = -A, \quad \beta = -B + A^2, \quad \gamma = -C + A^2, \quad \delta = -D + (C + B)A - A^3.$$

According to [**12, Corollary 7.2**], a GL(2)-structure is torsion-free if and only if

$$\left[V(\lambda), \frac{d}{d\lambda}V(\lambda)\right] \in \mathrm{span}\left\{V(\lambda), \frac{d}{d\lambda}V(\lambda), \frac{d^2}{d\lambda^2}V(\lambda)\right\}, \tag{9}$$

for any $\lambda \in \mathbb{R}$. This, due to [**12, Corollary 7.4**] applied to the framing

$$(V_0, 3V_1, 3V_2, V_3),$$

is expressed as eight linear equations for structural functions c_{ij}^k defined by $[V_i, V_j] = \sum_k c_{ij}^k V_k$. However, in the present case, the vector fields V_i are special and four equations are void. Indeed, the nontrivial equations are as follows:

$$c_{23}^0 = 0, \qquad c_{23}^1 - 2c_{13}^0 = 0,$$
$$c_{23}^2 - 2c_{13}^1 + c_{03}^0 + 3c_{12}^0 = 0, \qquad c_{23}^3 - 2c_{13}^2 + c_{03}^1 + 3c_{12}^1 - 2c_{02}^0 = 0$$

(the equations differ from equations in [**12**] because of the factor 3 next to V_1 and V_2 in the present paper). Substituting the structural functions, which can be easily computed, we get the system (8).

Now, we consider

$$L_0 = V(\lambda) - \left(\lambda - \frac{1}{3}A\right)\frac{d}{d\lambda}V(\lambda) \mod \partial_\lambda$$

and

$$L_1 = \frac{1}{3}\frac{d}{d\lambda}V(\lambda) \mod \partial_\lambda.$$

Due to (9), the commutator $[L_0, L_1]$ lies in the span of $\{L_0, L_1, \frac{d^2}{d\lambda^2}V(\lambda)\}$ mod ∂_λ. Moreover, since

$$L_0 = \partial_3 \mod \partial_1, \partial_0, \partial_\lambda \quad \text{and} \quad L_1 = \partial_2 \mod \partial_1, \partial_0, \partial_\lambda,$$

we get $[L_0, L_1] = \varphi\frac{d^2}{d\lambda^2}V(\lambda) \mod \partial_\lambda$ for some φ. One checks by direct computations that $\mu(\lambda)$ and $\nu(\lambda)$ are chosen such that $\varphi = 0$ and the coefficient of $[L_0, L_1]$ next to ∂_λ vanishes as well. □

References

[1] **M. Berger**, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes, *Bull. Soc. Math. France* **83** (1955), 279–330. 1

[2] **R. L. Bryant**, Two exotic holonomies in dimension four, path geometries, and twistor theory, in *Complex geometry and Lie theory*, *Proc. Sympos. Pure Math.* **53**, Amer. Math. Soc., Providence, RI, 1991, pp. 33–88. 1, 7, 11

[3] **R. L. Bryant**, **P. A. Griffiths**, Characteristic cohomology of differential systems. II. Conservation laws for a class of parabolic equations, *Duke Math. J.* **78** (1995), 531–676. 3

[4] **M. Dunajski**, **E. V. Ferapontov**, **B. Kruglikov**, On the Einstein-Weyl and conformal self-duality equations, *J. Math. Phys.* **56** (2015), 083501, 10. arXiv:1406.0018 7

[5] **M. Dunajski**, **W. Kryński**, Einstein-Weyl geometry, dispersionless Hirota equation and Veronese webs, *Math. Proc. Cambridge Philos. Soc.* **157** (2014), 139–150. arXiv:1301.0621 11

[6] **M. Dunajski**, **P. Tod**, Paraconformal geometry of nth-order ODEs, and exotic holonomy in dimension four, *J. Geom. Phys.* **56** (2006), 1790–1809. arXiv:math/0502524 7

[7] **E. V. Ferapontov**, **B. Kruglikov**, Dispersionless integrable hierarchies and $\mathrm{GL}(2, \mathbb{R})$ geometry, 2016. arXiv:1607.01966 2, 9

[8] **E. V. Ferapontov**, **B. S. Kruglikov**, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, *J. Differential Geom.* **97** (2014), 215–254. arXiv:1208.2728 2

[9] **M. Godlinski**, **P. Nurowski**, $\mathrm{GL}(2, \mathbb{R})$ geometry of ODE's, *J. Geom. Phys.* **60** (2010), 991–1027. 7

[10] **N. Hitchin**, Complex manifolds and Einstein's equations, in *Twistor geometry and nonlinear systems*, *Lecture Notes in Math.* **970**, Springer, Berlin, 1982, pp. 73–99. 1

[11] **T. A. Ivey**, **J. M. Landsberg**, *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, *Graduate Studies in Mathematics* **61**, American Mathematical Society, Providence, RI, 2003. 3

[12] **W. Kryński**, Paraconformal structures and differential equations, *Differential Geom. Appl.* **28** (2010), 523–531. 7, 8, 11, 12

[13] **W. Kryński**, Paraconformal structures, ordinary differential equations and totally geodesic manifolds, *J. Geom. Phys.* **103** (2016), 1–19. 8, 11

[14] **W. Kryński**, On deformations of the dispersionless Hirota equation, *J. Geom. Phys.* **127** (2018), 46–54. 11

[15] **S. V. Manakov**, **P. M. Santini**, Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking, *J. Phys. A* **44** (2011), 345203, 19. arXiv:1011.2619 9

[16] **S. V. Manakov**, **P. M. Santini**, Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields, *Journal of Physics: Conference Series* **482** (2014), 012029. arXiv:1312.2740 9

[17] **P. Nurowski**, Comment on $\mathbf{GL(2,\mathbb{R})}$ geometry of fourth-order ODEs, *J. Geom. Phys.* **59** (2009), 267–278. 7

[18] **A. D. Smith**, Integrable GL(2) geometry and hydrodynamic partial differential equations, *Comm. Anal. Geom.* **18** (2010), 743–790. arXiv:0912.2789 7

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