published version: Adv. Math. 354 (2019), Article ID 106747, 36 pp.

Minimal Lagrangian Connections on Compact Surfaces

THOMAS METTLER

ABSTRACT. We introduce the notion of a minimal Lagrangian connection on the tangent bundle of a manifold and classify all such connections in the case where the manifold is a compact oriented surface of non-vanishing Euler characteristic. Combining our classification with results of Labourie and Loftin, we conclude that every properly convex projective surface arises from a unique minimal Lagrangian connection.

1. Introduction

1.1. Background

A *projective manifold* is a pair $(M, \mathfrak{p})$ consisting of a smooth manifold M and a *projective structure* $\mathfrak{p}$, that is, an equivalence class of torsion-free connections on the tangent bundle TM, where two such connections are called *projectively equivalent* if they share the same geodesics up to parametrisation. A projective manifold $(M, \mathfrak{p})$ is called *properly convex* if it arises as a quotient of a properly convex open set $\tilde{M} \subset \mathbb{RP}^n$ by a group $\Gamma \subset \mathrm{PSL}(n+1, \mathbb{R})$ of projective transformations which acts discretely and properly discontinuously. The geodesics of $\mathfrak{p}$ are the projections to $M = \Gamma \backslash \tilde{M}$ of the projective line segments contained in $\tilde{M}$. In particular, locally the geodesics of a properly convex projective structure $\mathfrak{p}$ can be mapped diffeomorphically to segments of straight lines, that is, $\mathfrak{p}$ is *flat*.

It follows from the work of Cheng–Yau [9, 10] that the universal cover $\tilde{M}$ of a properly convex projective manifold $(M, \mathfrak{p})$ determines a unique properly embedded hyperbolic affine sphere $f : \tilde{M} \to \mathbb{R}^{n+1}$, which is asymptotic to the cone over $\tilde{M}$ in $\mathbb{R}^{n+1}$. The *Blaschke metric* and *Blaschke connection* induced by f descend to the quotient $\Gamma \backslash \tilde{M}$ and equip M with a complete Riemannian metric g and projectively flat connection $\nabla \in \mathfrak{p}$, see the work of Loftin [30]. The difference between ∇ and the Levi-Civita connection of the Blaschke metric is encoded in terms of a cubic form, the so-called *Fubini–Pick form* of f. For an introduction to affine differential geometry the reader may consult [33] as well as [29] for a nice survey on affine spheres.

Properly convex projective surfaces are of particular interest, as they may be seen – through the work of Hitchin [25], Goldman [20] and Choi–Goldman [12] – as the natural generalisation of the notion of a hyperbolic Riemann surface. In the case of a properly convex oriented surface $(\Sigma, \mathfrak{p})$, the Fubini–Pick form is the real part of a cubic differential that is holomorphic with respect to the Riemann surface structure

Date: March 4, 2019.

on Σ defined by the orientation and the conformal equivalence class of the Blaschke metric. Conversely, Wang [36] observed that a holomorphic cubic differential C on a closed hyperbolic Riemann surface $(\Sigma, [g])$ determines a unique conformal Riemannian metric g whose Gauss curvature K_g satisfies

$$K_g = -1 + 2|C|_g^2, \tag{1.1}$$

where $|C|_g$ denotes the point-wise tensor norm of C with respect to the Hermitian metric induced by g on the third power of the canonical bundle of Σ. Furthermore, the pair $(g, \mathrm{Re}(C))$ can be realized as the Blaschke metric and Fubini–Pick form of a complete hyperbolic affine sphere $f : \tilde{M} \to \mathbb{R}^3$ defined on the universal cover $\tilde{M}$ of M. In particular, combining Wang's work with the work of Loftin establishes – on a compact oriented surface of negative Euler characteristic – a bijective correspondence between properly convex projective structures and pairs $([g], C)$ consisting of a conformal structure $[g]$ and a cubic holomorphic differential C, see [30]. This correspondence was also discovered independently by Labourie [26]. Since then, Benoist–Hulin [1] have extended the correspondence to noncompact projective surfaces with finite Finsler volume and Dumas–Wolf [14] study the case of polynomial cubic differentials on the complex plane.

In [28], Libermann constructs a para-Kähler structure (h_0, Ω_0) on the open submanifold $A_0 \subset \mathbb{RP}^n \times \mathbb{RP}^{n*}$ consisting of non-incident point-line pairs. A para-Kähler structure may be thought of as a split-complex analogue of the notion of a Kähler structure. In particular, h_0 is a pseudo-Riemannian metric of split-signature (n, n) and Ω_0 a symplectic form, so that there exists an endomorphism of the tangent bundle relating h_0 and Ω_0 which squares to become the identity map. In [22, 23], Hildebrand – see also the related work [17, 19, 35] – observed that proper affine spheres $f : M \to \mathbb{R}^{n+1}$ correspond to minimal Lagrangian immersions $\hat{f} : M \to A_0$. Thus, the result of Hildebrand, combined with the work of Cheng–Yau, associates a minimal Lagrangian immersion to every properly convex projective manifold.

1.2. Minimal Lagrangian connections

Here we propose a generalization of the notion of a properly convex projective surface which arises naturally from the concept of a *minimal Lagrangian connection*. In joint work with Dunajski the author has shown that the construction of Libermann is a special case of a more general result: In [15], it is shown that a projective structure $\mathfrak{p}$ on an n-manifold M canonically defines an almost para–Kähler structure $(h_\mathfrak{p}, \Omega_\mathfrak{p})$ on the total space of a certain affine bundle $A \to M$, whose underlying vector bundle is the cotangent bundle of M. The bundle $A \to M$ has the crucial property that its sections are in one-to-one correspondence with the representative connections of $\mathfrak{p}$. Therefore, fixing a representative connection $\nabla \in \mathfrak{p}$ gives a section $s_\nabla : M \to A$ and hence an isomorphism $\psi_\nabla : T^*M \to A$, by declaring the origin of the affine fibre A_p to be $s_\nabla(p)$ for all $p \in M$. Correspondingly, we obtain a pair $(h_\nabla, \Omega_\nabla) = \psi_\nabla^*(h_\mathfrak{p}, \Omega_\mathfrak{p})$ on the total space of the cotangent bundle. Besides being a geometric structure of interest in itself (see [15] for details), the pair $(h_\nabla, \Omega_\nabla)$ has the natural property

$$o^* h_\nabla = (s_\nabla)^* h_\mathfrak{p} = -\left(\frac{1}{n-1}\right) \mathrm{Ric}^+(\nabla)$$

and

$$o^* \Omega_\nabla = (s_\nabla)^* \Omega_\mathfrak{p} = \left(\frac{1}{n+1}\right) \mathrm{Ric}^-(\nabla),$$

where $o : M \to T^*M$ denotes the zero-section and $\mathrm{Ric}^\pm(\nabla)$ the symmetric (respectively, the anti-symmetric) part of the Ricci curvature $\mathrm{Ric}(\nabla)$ of ∇. Consequently, we call ∇ *Lagrangian* if the Ricci tensor of ∇ is symmetric, or equivalently, if the zero-section o is a Lagrangian submanifold of (T^*M, Ω_∇). Likewise, we call ∇ *timelike/spacelike* if $\pm\mathrm{Ric}^+(\nabla)$ is positive definite, or equivalently, if the zero-section o is a timelike/spacelike submanifold of (T^*M, h_∇). The upper sign corresponds to the timelike case and lower sign to the spacelike case. Moreover, we call ∇ *minimal* if the zero-section is a minimal submanifold of (T^*M, h_∇).

We henceforth restrict our considerations to the case of oriented surfaces. We show (see Theorem 4.4 below) that a timelike/spacelike Lagrangian connection ∇ is minimal if and only if

$$R^{ij}\left(2\nabla_i R_{jk} - \nabla_k R_{ij}\right) = 0,$$

where R_{ij} denotes the Ricci tensor of ∇ and R^{ij} its inverse. We then show that a minimal Lagrangian connection ∇ on an oriented surface Σ defines a triple (g, β, C) on Σ, consisting of a Riemannian metric g, a 1-form β and a cubic differential C, so that the following equations hold

$$K_g = \pm 1 + 2\,|C|_g^2 + \delta_g \beta, \qquad \bar\partial C = \left(\beta - \mathrm{i} \star_g \beta\right) \otimes C, \qquad \mathrm{d}\beta = 0. \tag{1.2}$$

As usual, $\mathrm{i} = \sqrt{-1}$, $\bar\partial$ denotes the "del-bar" operator with respect to the integrable almost complex structure J induced on Σ by $[g]$ and the orientation, $\star_g$, δ_g and K_g denote the Hodge-star, co-differential and Gauss curvature with respect to g, respectively. Recall that $|C|_g$ denotes the point-wise tensor norm of C with respect to the Hermitian metric induced by g on the third power of the canonical bundle of Σ. As a consequence, we use a result of Labourie [26] to prove that if ∇ is a spacelike minimal Lagrangian connection on a compact oriented surface Σ defining a flat projective structure $\mathfrak{p}(\nabla)$, then $(\Sigma, \mathfrak{p})$ is a properly convex projective surface. Moreover, the zero-section is a totally geodesic submanifold of $(T^*\Sigma, h_\nabla)$ if and only if ∇ is the Levi-Civita connection of a hyperbolic metric.

We also show that a minimal Lagrangian connection defines a flat projective structure if and only if β vanishes identically. In particular, we recover Wang's equation (1.1) in the projectively flat case. In the projectively flat case Labourie [26] interpreted the first two equations as an instance of Hitchin's Higgs bundle equations [24]. In the case with $\beta \neq 0$ the above triple of equations falls into the general framework of *symplectic vortex equations* [13] (see also [3]). Furthermore, it appears likely that the above equations also admit an interpretation in terms of affine differential geometry, but this will be addressed elsewhere.

The last two of the equations (1.2) say that locally there exists a (real-valued) function r so that $\mathrm{e}^{-2r} C$ is holomorphic. As a consequence of this we show that the only examples of minimal Lagrangian connections on the 2-sphere are Levi-Civita connections of metrics of positive Gauss curvature.

Furthermore, if $(\Sigma, [g])$ is a compact Riemann surface of negative Euler characteristic $\chi(\Sigma)$, then the metric g of the triple (g, β, C) is uniquely determined in terms of $([g], \beta, C)$. This leads to a quasi-linear elliptic PDE of vortex type, which belongs to a class of equations solved in [17] using the technique of sub – and supersolutions (see also [14] for the case when β vanishes identically). Here instead, we use the calculus of variations and prove existence and uniqueness of a smooth minimum of the following functional defined on the Sobolev space $W^{1,2}(\Sigma)$

$$\mathcal{E}_{\kappa,\tau} : W^{1,2}(\Sigma) \to \overline{\mathbb{R}}, \quad u \mapsto \frac{1}{2}\int_\Sigma |\mathrm{d}u|^2_{g_0} - 2u - \kappa \mathrm{e}^{2u} + \tau \mathrm{e}^{-4u} dA_{g_0},$$

where $\kappa, \tau \in C^\infty(\Sigma)$ satisfy $\kappa < 0$, $\tau \geqslant 0$ and g_0 denotes the hyperbolic metric in the conformal equivalence class $[g]$.

An immediate consequence of (1.2) is that the area of a spacelike minimal Lagrangian connection ∇ – by which we mean the area of $o(\Sigma) \subset (T^*\Sigma, h_\nabla)$ – satisfies the inequality

$$\mathrm{Area}(\nabla) \geqslant -2\pi\chi(\Sigma).$$

Therefore, we call a spacelike minimal Lagrangian connection with area $-2\pi\chi(\Sigma)$ *area minimising*. We obtain:

Theorem A. *Let Σ be a compact oriented surface with $\chi(\Sigma) < 0$. Then we have:*

(i) *there exists a one-to-one correspondence between area minimising Lagrangian connections on $T\Sigma$ and pairs $([g], \beta)$ consisting of a conformal structure $[g]$ and a closed 1-form β on Σ;*

(ii) *there exists a one-to-one correspondence between non-area minimising minimal Lagrangian connections on $T\Sigma$ and pairs $([g], C)$ consisting of a conformal structure $[g]$ and a non-trivial cubic differential C on Σ that satisfies $\overline{\partial} C = \left(\beta - \mathrm{i} \star_g \beta\right) \otimes C$ for some closed 1-form β.*

Using the classification of properly convex projective structures by Loftin [30] and Labourie [26], it follows that every properly convex projective structure on Σ arises from a unique spacelike minimal Lagrangian connection, paralleling the result of Hildebrand.

1.3. Related work

After the first version of this article appeared on the arXiv, Daniel Fox informed the author about his interesting paper [17], which contains some closely related results. Here we briefly compare our results which were arrived at independently. In a previous preprint [16] (see also [18]), Fox introduced the notion of an AH (affine hypersurface) structure which is a pair comprising a projective structure and a conformal structure satisfying a compatibility condition which is automatic in two dimensions. He then proceeds to postulate Einstein equations for AH structures, which are motivated by Calderbank's work on Einstein–Weyl structures on surfaces [5]. Subsequently in [17], Fox classifies the Einstein AH structures on closed oriented surfaces and in particular, observes that in the case of negative Euler-characteristic they precisely correspond to the properly convex projective structures.

On the 2-sphere S^2 he recovers the Einstein–Weyl structures of Calderbank. On S^2, the Einstein AH structures and the minimal Lagrangian connections "overlap" in the space of metrics of constant positive Gauss–curvature. In the case of negative Euler characteristic, the minimal Lagrangian connections are however *strictly* more general than the Einstein AH structures. Indeed, in this case, the projective structures arising from minimal Lagrangian connections provide a new and previously unstudied class of projective structures.

This new class of (possibly) curved projective structures may be thought of as a generalization of the notion of a (flat) properly convex projective structure. One would expect that this class exhibits interesting properties, similar to those of properly convex projective structures. As a first result in this direction, it is shown in [32], that the geodesics of a minimal Lagrangian connection naturally give rise to a flow admitting a dominated splitting (a certain weakening of the notion of an Anosov flow). In particular, this flow provides a generalization of the geodesic flow induced by the Hilbert metric on the quotient surface of a divisible convex set.

Furthermore, in joint work in progress by the author and A. Cap [8], the notion of a minimal Lagrangian connection is extended to all so-called $|1|$-graded parabolic geometries. This is a class of geometric structures which, besides projective geometry, includes (but is not restricted to) conformal geometry, (almost) Grassmannian geometry and (almost) quaternionic geometry.

Acknowledgements

The author is grateful to Andreas Cap, Daniel Fox, Gabriel Paternain, Nigel Hitchin, Norbert Hungerbühler, Tobias Weth and Luca Galimberti for helpful conversations or correspondence. The author also would like to thank the anonymous referees for several helpful suggestions.

2. Preliminaries

Throughout the article Σ will denote an oriented smooth 2-manifold without boundary. All manifolds and maps are assumed to be smooth and we adhere to the convention of summing over repeated indices.

2.1. The coframe bundle

We denote by $\upsilon : F \to \Sigma$ the bundle of orientation preserving coframes whose fibre at $p \in \Sigma$ consists of the linear isomorphisms $f : T_p\Sigma \to \mathbb{R}^2$ that are orientation preserving with respect to the fixed orientation on Σ and the standard orientation on $\mathbb{R}^2$. Recall that $\upsilon : F \to \Sigma$ is a principal right $\mathrm{GL}^+(2,\mathbb{R})$-bundle with right action defined by the rule $R_a(f) = f \cdot a = a^{-1} \circ f$ for all $a \in \mathrm{GL}^+(2,\mathbb{R})$. The bundle F is equipped with a tautological $\mathbb{R}^2$-valued 1-form $\omega = (\omega^i)$ defined by $\omega_f = f \circ \upsilon'_f$, and this 1-form satisfies the equivariance property $R_a^*\omega = a^{-1}\omega$. A torsion-free connection ∇ on $T\Sigma$ corresponds to a $\mathfrak{gl}(2,\mathbb{R})$-valued connection 1-form $\theta = (\theta^i_j)$ on F satisfying the structure equations

$$\mathrm{d}\omega = -\theta \wedge \omega, \tag{2.1}$$

$$\mathrm{d}\theta = -\theta \wedge \theta + \Theta, \tag{2.2}$$

where Θ denotes the curvature 2-form of θ. The Ricci curvature of ∇ is the (not necessarily symmetric) covariant 2-tensor field $\mathrm{Ric}(\nabla)$ on Σ satisfying

$$\mathrm{Ric}(\nabla)(X,Y) = \mathrm{tr}\left(Z \mapsto \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y\right), \quad Z \in \Gamma(TM),$$

for all vector fields X, Y on Σ. Denoting by $\mathrm{Ric}^{\pm}(\nabla)$ the symmetric (respectively, the anti-symmetric) part of the Ricci curvature of ∇, so that $\mathrm{Ric}(\nabla) = \mathrm{Ric}^+(\nabla) + \mathrm{Ric}^-(\nabla)$, the (projective) Schouten tensor of ∇ is defined as

$$\mathrm{Schout}(\nabla) = \mathrm{Ric}^+(\nabla) - \frac{1}{3}\mathrm{Ric}^-(\nabla).$$

Since the components of ω are a basis for the υ-semibasic forms on F,[1] it follows that there exist real-valued functions S_{ij} on F such that

$$\upsilon^*\mathrm{Schout}(\nabla) = \omega^t S\omega = S_{ij}\omega^i \otimes \omega^j,$$

where $S = (S_{ij})$. Note that

$$R_a^* S = a^t S a$$

for all $a \in \mathrm{GL}^+(2,\mathbb{R})$, since $\omega^t S\omega$ is invariant under R_a. In terms of the functions S_{ij} the curvature 2-form $\Theta = (\Theta^i_j)$ can be written as[2]

$$\Theta^i_j = \left(\delta^i_{[k}S_{l]j} - \delta^i_j S_{[kl]}\right)\omega^k \wedge \omega^l, \tag{2.3}$$

or explicitly

$$\Theta = \begin{pmatrix} 2S_{21} - S_{12} & S_{22} \\ -S_{11} & S_{21} - 2S_{12} \end{pmatrix} \omega^1 \wedge \omega^2. \tag{2.4}$$

2.2. The orthonormal coframe bundle

Recall that if g is a Riemannian metric on the oriented surface Σ, the Levi-Civita connection (φ^i_j) of g is the unique connection on the coframe bundle $\upsilon : F \to \Sigma$ satisfying

$$\mathrm{d}\omega^i = -\varphi^i_j \wedge \omega^j,$$
$$\mathrm{d}g_{ij} = g_{ik}\varphi^k_j + g_{kj}\varphi^k_i,$$

where we write $\upsilon^*g = g_{ij}\omega^i \otimes \omega^j$ for real-valued functions $g_{ij} = g_{ji}$ on F. Differentiating these equations implies that there exists a unique function K_g, the Gauss curvature of g, so that

$$\mathrm{d}\varphi^i_j + \varphi^i_k \wedge \varphi^k_j = g_{jk}K_g\omega^i \wedge \omega^k.$$

We may reduce F to the SO(2)-subbundle F_g consisting of orientation preserving coframes that are also orthonormal with respect to g , that is, the bundle defined by the equations $g_{ij} = \delta_{ij}$. On F_g the identity $\mathrm{d}g_{ij} = 0$ implies the identities

[1] Recall that a 1-form $\alpha \in \Omega^1(M)$ is semibasic for the projection $\pi : M \to N$ if α vanishes on vector fields that are tangent to the π-fibres.

[2] For a matrix $S = (S_{ij})$ we denote by $S_{(ij)}$ its symmetric part and by $S_{[ij]}$ its anti-symmetric part, so that $S_{ij} = S_{(ij)} + S_{[ij]}$.

$\varphi_1^1 = \varphi_2^2 = 0$ as well as $\varphi_2^1 + \varphi_1^2 = 0$. Therefore, writing $\varphi := \varphi_1^2$, we obtain the structure equations

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_2 \wedge \varphi, \\ \mathrm{d}\omega_2 &= -\varphi \wedge \omega_1, \\ \mathrm{d}\varphi &= -K_g \omega_1 \wedge \omega_2, \end{aligned} \tag{2.5}$$

where $\omega_i = \delta_{ij}\omega^j$. Continuing to denote the basepoint projection $F_g \to \Sigma$ by υ, the area form dA_g of g satisfies $\upsilon^* dA_g = \omega_1 \wedge \omega_2$. Also, note that a complex-valued 1-form α on Σ is a (1,0)-form for the complex structure J induced on Σ by g and the orientation if and only if $\upsilon^*\alpha$ is a complex multiple of the complex-valued form $\omega = \omega_1 + \mathrm{i}\omega_2$. In particular, denoting by K_Σ the canonical bundle of Σ with respect to J, a section A of the ℓ-th tensorial power of K_Σ satisfies $\upsilon^* A = a\omega^\ell$ for some unique complex-valued function a on F_g. Denote by $S^3_0(T^*\Sigma)$ the trace-free part of $S^3(T^*\Sigma)$ with respect to $[g]$, where $S^3(T^*\Sigma)$ denotes the third symmetrical power of the cotangent bundle of Σ. The proof of the following lemma is an elementary computation and thus omitted.

Lemma 2.1. *Suppose $W \in \Gamma\left(S^3_0(T^*\Sigma)\right)$. Then there exists a unique cubic differential $C \in \Gamma(K^3_\Sigma)$ so that $\mathrm{Re}(C) = W$. Moreover, writing $\upsilon^* W = w_{ijk}\omega_i \otimes \omega_j \otimes \omega_k$ for unique real-valued functions w_{ijk} on F_g, totally symmetric in all indices, the cubic differential satisfies $\upsilon^* C = (w_{111} + \mathrm{i}w_{222})\omega^3$.*

In complex notation, the structure equations of a cubic differential $C \in \Gamma(K^3_\Sigma)$ can be written as follows. Writing $\upsilon^* C = c\omega^3$ for a complex-valued function c on F_g, it follows from the SO(2)-equivariance of $c\omega^3$ that there exist complex-valued functions c' and c'' on F_g such that

$$\mathrm{d}c = c'\omega + c''\overline{\omega} + 3\mathrm{i}c\varphi,$$

where we write $\overline{\omega} = \omega_1 - \mathrm{i}\omega_2$. Note that the Hermitian metric induced by g on K^3_Σ has Chern connection D given by

$$c \mapsto \mathrm{d}c - 3\mathrm{i}c\varphi.$$

In particular, the (0,1)-derivative of C with respect to D is represented by c'', that is, $\upsilon^*(\mathrm{D}^{0,1}C) = c''\omega^3 \otimes \overline{\omega}$. Since $\overline{\partial} = \mathrm{D}^{0,1}$, we obtain

$$\upsilon^*\left(\overline{\partial} C\right) = c''\omega^3 \otimes \overline{\omega}. \tag{2.6}$$

Also, we record the identity

$$\upsilon^*|C|^2_g = |c|^2.$$

Moreover, recall that for $u \in C^\infty(\Sigma)$ we have the following standard identity for the change of the Gauss curvature of a metric g under conformal rescaling

$$K_{\mathrm{e}^{2u}g} = \mathrm{e}^{-2u}\left(K_g - \Delta_g u\right),$$

where $\Delta_g = -\left(\delta_g \mathrm{d} + \mathrm{d}\delta_g\right)$ is the negative of the Laplace–Beltrami operator with respect to g. Also,

$$dA_{\mathrm{e}^{2u}g} = \mathrm{e}^{2u} dA_g$$

for the change of the area form dA_g,

$$\Delta_{\mathrm{e}^{2u}g} = \mathrm{e}^{-2u}\Delta_g$$

for Δ_g acting on functions and

$$\delta_{e^{2u}g} = e^{-2u}\delta_g$$

for the co-differential acting on 1-forms. Finally, the norm of C changes as

$$|C|^2_{e^{2u}g} = e^{-6u}|C|^2_g.$$

2.3. The cotangent bundle and induced structures

Recall that we have a $\mathrm{GL}^+(2,\mathbb{R})$ representation ϱ on $\mathbb{R}_2$ – the real vector space of row vectors of length two with real entries – defined by the rule $\varrho(a)\xi = \xi a^{-1}$ for all $\xi \in \mathbb{R}_2$ and $a \in \mathrm{GL}^+(2,\mathbb{R})$. The cotangent bundle of Σ is the vector bundle associated to the coframe bundle F via the representation ϱ, that is, the bundle obtained by taking the quotient of $F \times \mathbb{R}_2$ by the $\mathrm{GL}^+(2,\mathbb{R})$-right action induced by ϱ. Consequently, a 1-form on Σ is represented by an $\mathbb{R}_2$-valued function ξ on F which is $\mathrm{GL}^+(2,\mathbb{R})$-equivariant, that is, ξ satisfies $R_a^*\xi = \xi a$ for all $a \in \mathrm{GL}^+(2,\mathbb{R})$.

Using θ we may define a Riemannian metric h_∇ as well as a symplectic form Ω_∇ on $T^*\Sigma$ as follows. Let

$$\pi : F \times \mathbb{R}_2 \to (F \times \mathbb{R}_2)/\mathrm{GL}^+(2,\mathbb{R}) \simeq T^*\Sigma$$

denote the quotient projection. Writing

$$\psi = \mathrm{d}\xi - \xi\theta - \xi\omega\xi - \omega^t S^t$$

or in components $\psi = (\psi_i)$ with

$$\psi_i = \mathrm{d}\xi_i - \xi_j\theta^j_i - \xi_j\omega^j\xi_i - S_{ji}\omega^j,$$

we consider the covariant 2-tensor field $T_\nabla = \psi\omega := \psi_i \otimes \omega^i$. Note that the π-semibasic 1-forms on $F \times \mathbb{R}_2$ are given by the components of ω and $\mathrm{d}\xi - \xi\theta$, or, equivalently, by the components of ω and ψ. Indeed, the components of ω are semibasic by the definition of ω. Moreover, if X_v for $v \in \mathfrak{gl}(2,\mathbb{R})$ is a fundamental vector field for the coframe bundle $F \to \Sigma$, that is, the vector field associated to the flow $R_{\exp(tv)}$, then

$$\begin{aligned}(\mathrm{d}\xi - \xi\theta)(X_v) &= -\xi\theta(X_v) + \lim_{t\to 0}\frac{1}{t}\left((R_{\exp(tv)})^*\xi - \xi\right)\\ &= -\xi v + \lim_{t\to 0}\frac{1}{t}\left(\xi\exp(tv) - \xi\right) = -\xi v + \xi v = 0,\end{aligned}$$

where we have used that the connection θ maps a fundamental vector field X_v to its generator $v \in \mathfrak{gl}(2,\mathbb{R})$. Since the fundamental vector fields span the vector fields tangent to fibres of π, it follows that the components of $\mathrm{d}\xi - \xi\theta$ are π-semibasic. Moreover,

$$R_a^*\psi = \mathrm{d}\xi\, a - \xi a a^{-1}\theta a - \xi a a^{-1}\omega\xi a - \omega^t(a^{-1})^t a^t S^t a = \psi a, \tag{2.7}$$

$$R_a^*\omega = a^{-1}\omega, \tag{2.8}$$

for all $a \in \mathrm{GL}^+(2,\mathbb{R})$, it follows that the π-semibasic tensor field T_∇ is invariant under the $\mathrm{GL}^+(2,\mathbb{R})$-right action and hence there exists a unique symmetric covariant 2-tensor field h_∇ and a unique anti-symmetric covariant 2-tensor field Ω_∇ on

$T^*\Sigma$ such that

$$\pi^*\left(h_\nabla + \Omega_\nabla\right) = T_\nabla.$$

Using the structure equation (2.1), we compute

$$\begin{aligned} \pi^*\Omega_\nabla &= \mathrm{d}\xi_i \wedge \omega^i - \xi_j \theta^j_i \wedge \omega^i - \xi_i \xi_j \omega^j \wedge \omega^i - S_{ji}\omega^j \wedge \omega^i \\ &= \mathrm{d}\xi_i \wedge \omega^i + \xi_i \mathrm{d}\omega^i - S_{ij}\omega^i \wedge \omega^j \\ &= \mathrm{d}(\xi_i \omega^i) - S_{[ij]}\omega^i \wedge \omega^j. \end{aligned}$$

The 1-form $\xi\omega = \xi_i\omega^i$ on $F \times \mathbb{R}_2$ is π-semibasic and R_a invariant, hence the π-pullback of a unique 1-form τ on $T^*\Sigma$ which is the tautological 1-form of $T^*\Sigma$. Recall that the canonical symplectic form on $T^*\Sigma$ is $\Omega_0 = \mathrm{d}\tau$, hence Ω_∇ defines a symplectic structure on $T^*\Sigma$ which is the canonical symplectic structure twisted with the (closed) 2-form $\frac{1}{3}\mathrm{Ric}^-(\nabla)$

$$\Omega_\nabla = \Omega_0 + \Upsilon^*\left(\frac{1}{3}\mathrm{Ric}^-(\nabla)\right),$$

where $\Upsilon : T^*\Sigma \to \Sigma$ denotes the basepoint projection. In particular, denoting by $o : \Sigma \to T^*\Sigma$ the zero Υ-section, the definition of the Schouten tensor gives

$$o^*\Omega_\nabla = \frac{1}{3}\mathrm{Ric}^-(\nabla).$$

This shows:

Proposition 2.2. *The zero section of $T^*\Sigma$ is a Ω_∇-Lagrangian submanifold if and only if ∇ has symmetric Ricci tensor.*

Which motivates:

Definition 2.3. A torsion-free connection ∇ on $T\Sigma$ is called *Lagrangian* if $\mathrm{Ric}^-(\nabla)$ vanishes identically.

Also, we obtain for the symmetric part

$$\pi^*h_\nabla = \psi \circ \omega := \psi_1 \circ \omega^1 + \psi_2 \circ \omega^2$$

where $\circ$ denotes the symmetric tensor product. Since the four 1-forms $\psi_1, \psi_2, \omega^1, \omega^2$ are linearly independent, it follows that h_∇ is non-degenerate and hence defines a pseudo-Riemannian metric of split signature $(1, 1, -1, -1)$ on $T^*\Sigma$.

Remark 2.4. The motivation for introducing the pair $(h_\nabla, \Omega_\nabla)$ is its projective invariance, i.e., suitably interpreted, the pair $(h_\nabla, \Omega_\nabla)$ does only depend on the projective equivalence class of the connection ∇. Moreover, the metric h_∇ is anti-self-dual and Einstein. We refer the reader to [15] as well as [6] for further details.

From the definition of the Schouten tensor and h_∇ we immediately obtain

$$o^*h_\nabla = -\mathrm{Ric}^+(\nabla). \tag{2.9}$$

Following standard pseudo-Riemannian submanifold theory, we call a tangent vector v *timelike* if $h_\nabla(v, v) < 0$ and *spacelike* if $h_\nabla(v, v) > 0$. Thus (2.9) motivates:

Definition 2.5. A torsion-free connection ∇ on $T\Sigma$ is called *timelike* if $\mathrm{Ric}^+(\nabla)$ is positive definite and *spacelike* if $\mathrm{Ric}^+(\nabla)$ is negative definite.

3. Twisted Weyl connections

We will see that timelike/spacelike minimal Lagrangian connections are *twisted Weyl connections*. In this section we study some properties of this class of connections that we will need later during the classification of spacelike minimal Lagrangian connections.

Let $[g]$ be a conformal structure on the smooth oriented surface Σ. By a $[g]$-Weyl connection on Σ we mean a torsion-free connection on $T\Sigma$ preserving the conformal structure $[g]$. It follows from Koszul's identity that a $[g]$-Weyl connection can be written in the following form

$$ {}^{(g,\beta)}\nabla = {}^{g}\nabla + g \otimes \beta^{\sharp} - \beta \otimes \mathrm{Id} - \mathrm{Id} \otimes \beta, $$

where $g \in [g]$, $\beta \in \Omega^1(\Sigma)$ is a 1-form and $\beta^{\sharp}$ denotes the g-dual vector field to β. We will use the notation ${}^{[g]}\nabla$ to denote a general $[g]$-Weyl connection.

Definition 3.1. A *twisted Weyl connection* ∇ on $(\Sigma, [g])$ is a connection on the tangent bundle of Σ which can be written as $\nabla = {}^{[g]}\nabla + \alpha$ for some $[g]$-Weyl connection ${}^{[g]}\nabla$ and some 1-form α with values in $\mathrm{End}(T\Sigma)$ satisfying the following properties:

(i) $\alpha(X)$ is trace-free and $[g]$-symmetric for all $X \in \Gamma(T\Sigma)$;
(ii) $\alpha(X)Y = \alpha(Y)X$ for all $X, Y \in \Gamma(T\Sigma)$.

Note that if α satisfies the above properties, then ${}^{[g]}\nabla + \alpha$ is torsion-free. Moreover, the covariant 3-tensor obtained by lowering the upper index of α with a metric $g \in [g]$ gives a section of $\Gamma(S^3_0(T^*\Sigma))$. Conversely, every $\mathrm{End}(T\Sigma)$-valued 1-form on Σ satisfying the above properties arises in this way. In other words, fixing a Riemannian metric $g \in [g]$ allows to identify the twist term α with a cubic differential.

Fixing a metric $g \in [g]$, the connection form $\theta = (\theta^i_j)$ of a twisted Weyl connection is given by

$$ \theta^i_j = \varphi^i_j + \left(b_k g^{ki} g_{jl} - \delta^i_j b_l - \delta^i_l b_j + a^i_{jl}\right)\omega^l, $$

where the map $(g_{ij}) : F \to S^2(\mathbb{R}_2)$ represents the metric g, the map $(b_i) : F \to \mathbb{R}_2$ represents the 1-form β and the map $(a^i_{jk}) : F \to \mathbb{R}^2 \otimes S^2(\mathbb{R}_2)$ represents the 1-form α. Moreover, (φ^i_j) denote the Levi-Civita connection forms of g. Reducing to the bundle F_g of g-orthonormal orientation preserving coframes, the connection form becomes

$$ \theta = \begin{pmatrix} -\beta & \star_g \beta - \varphi \\ \varphi - \star_g \beta & -\beta \end{pmatrix} + \begin{pmatrix} a^1_{11}\omega_1 + a^1_{12}\omega_2 & a^1_{12}\omega^1 + a^1_{22}\omega_2 \\ a^2_{11}\omega_1 + a^2_{12}\omega_2 & a^2_{12}\omega_1 + a^2_{22}\omega_2 \end{pmatrix}, $$

where we use the identity $\upsilon^*\left(\star_g \beta\right) = -b_2\omega_1 + b_1\omega_2$. By definition, on F_g the functions a^i_{jk} satisfy the identities

$$ a^i_{jk} = a^i_{kj}, \qquad a^k_{kj} = 0, \qquad \delta_{ki} a^k_{jl} = \delta_{kj} a^k_{il}. $$

Thus, writing $c_1 = a^1_{11}$ and $c_2 = a^2_{22}$, we obtain

$$ \theta = \begin{pmatrix} -\beta & \star_g \beta - \varphi \\ \varphi - \star_g \beta & -\beta \end{pmatrix} + \begin{pmatrix} c_1\omega_1 - c_2\omega_2 & -c_2\omega_1 - c_1\omega_2 \\ -c_2\omega_1 - c_1\omega_2 & -c_1\omega_1 + c_2\omega_2 \end{pmatrix}. $$

In order to compute the curvature form of θ we first recall that we write $\upsilon^*\beta = b_i\omega_i$ and since $b_i\omega_i$ is SO(2)-invariant, it follows that there exist unique real-valued functions b_{ij} on F_g such that

$$\begin{aligned} \mathrm{d}b_1 &= b_{11}\omega_1 + b_{12}\omega_2 + b_2\varphi, \\ \mathrm{d}b_2 &= b_{21}\omega_1 + b_{22}\omega_2 - b_1\varphi. \end{aligned}$$

Recall also that the area form of g satisfies $\upsilon^* dA_g = \omega_1 \wedge \omega_2$ and since $\star_g 1 = dA_g$, we get

$$\upsilon^*\delta_g\beta = -(b_{11} + b_{22}),$$

as well as

$$\upsilon^*\left(\mathrm{d}\star_g\beta\right) = (b_{11} + b_{22})\omega_1 \wedge \omega_2.$$

Since $c_1 + \mathrm{i}c_2$ represents a cubic differential on Σ, there exist unique real-valued functions c_{ij} on F_g such that

$$\begin{aligned} \mathrm{d}c_1 &= c_{11}\omega_1 + c_{12}\omega_2 - 3c_2\varphi, \\ \mathrm{d}c_2 &= c_{21}\omega_1 + c_{22}\omega_2 + 3c_1\varphi. \end{aligned}$$

Consequently, a straightforward calculation shows that the curvature form $\Theta = \mathrm{d}\theta + \theta \wedge \theta$ satisfies

(3.1)

$$\begin{aligned} \Theta = &\begin{pmatrix} -\mathrm{d}\beta & K_g dA_g + \mathrm{d}\star_g\beta - \frac{1}{2}|\alpha|^2_g\omega_1\wedge\omega_2 \\ -K_g dA_g - \mathrm{d}\star_g\beta + \frac{1}{2}|\alpha|^2_g\omega_1\wedge\omega_2 & -\mathrm{d}\beta \end{pmatrix} \\ &+ \begin{pmatrix} 2(b_1c_2 + b_2c_1) - (c_{12} + c_{21}) & 2(b_1c_1 - b_2c_2) + (c_{22} - c_{11}) \\ 2(b_1c_1 - b_2c_2) + (c_{22} - c_{11}) & 2(-b_1c_2 - b_2c_1) + (c_{12} + c_{21}) \end{pmatrix}\omega_1\wedge\omega_2, \end{aligned}$$

where we use the identity $\upsilon^*|\alpha|^2_g = 4\left((c_1)^2 + (c_2)^2\right)$.

3.1. A characterisation of twisted Weyl connections

We obtain a natural differential operator $\mathrm{D}_{[g]}$ acting on the space $\mathfrak{A}(\Sigma)$ of torsion-free connections on $T\Sigma$

$$\mathrm{D}_{[g]} : \mathfrak{A}(\Sigma) \to \Omega^2(\Sigma), \quad \nabla \mapsto \mathrm{tr}_g\,\mathrm{Ric}(\nabla)dA_g.$$

Note that this operator does indeed only depend on the conformal equivalence class of g. A twisted Weyl connection ∇ on $(\Sigma, [g])$ can be characterised by minimising the integral of $\mathrm{D}_{[g]}$ among its projective equivalence class $\mathfrak{p}(\nabla)$.

Proposition 3.2. *Suppose $\nabla' = {}^{[g]}\nabla + \alpha$ is a twisted Weyl connection on the compact Riemann surface $(\Sigma, [g])$. Then*

$$\inf_{\nabla\in\mathfrak{p}(\nabla')}\int_\Sigma \mathrm{D}_{[g]}(\nabla) = 4\pi\chi(\Sigma) - \|\alpha\|^2_g$$

and $4\pi\chi(\Sigma) - \|\alpha\|^2_g$ is attained precisely on ∇'.

Remark 3.3. Note that

$$\|\alpha\|^2_g = \int_\Sigma |\alpha|^2_g\, dA_g$$

does only depend on the conformal equivalence class of g.

Proof of Proposition 3.2. Write $\nabla' = {}^{(g,\beta)}\nabla + \alpha$ for some Riemannian metric $g \in [g]$, some 1-form β and some $\mathrm{End}(T\Sigma)$-valued 1-form α on Σ satisfying the properties of Definition 3.1. From (2.4) and the definition of the Schouten tensor it follows that

$$\upsilon^*\left(\mathrm{tr}_g \,\mathrm{Ric}(\nabla')dA_g\right) = \Theta^1_2 - \Theta^2_1,$$

where $\Theta = (\Theta^i_j)$ denotes the curvature form of ∇' pulled-back to F_g. Thus, equation (2.4) gives

$$\mathrm{tr}_g \,\mathrm{Ric}(\nabla')dA_g = 2K_g + 2\mathrm{d}\star_g \beta - |\alpha|^2_g dA_g \tag{3.2}$$

and hence

$$\int_\Sigma \mathrm{tr}_g \,\mathrm{Ric}(\nabla')dA_g = 4\pi\chi(\Sigma) - \|\alpha\|^2_g \tag{3.3}$$

by the Stokes and the Gauss–Bonnet theorem.

It is a classical result due to Weyl [37] that two torsion-free connections ∇^1, ∇^2 on $T\Sigma$ are projectively equivalent if and only if there exists a 1-form γ on Σ such that $\nabla^1 - \nabla^2 = \gamma \otimes \mathrm{Id} + \mathrm{Id} \otimes \gamma$. It follows that the connections in the projective equivalence class of ∇' can be written as

$$\nabla = \nabla' + \gamma \otimes \mathrm{Id} + \mathrm{Id} \otimes \gamma$$

with $\gamma \in \Omega^1(\Sigma)$. A simple computation gives

$$\mathrm{Ric}(\nabla) = \mathrm{Ric}(\nabla') + \gamma^2 - \mathrm{Sym}\,\nabla'\gamma + 3\,\mathrm{d}\gamma, \tag{3.4}$$

where $\mathrm{Sym} : \Gamma(T^*\Sigma \otimes T^*\Sigma) \to \Gamma(S^2(T^*\Sigma))$ denotes the natural projection. We compute

$$\begin{aligned}\mathrm{tr}_g \,\mathrm{Sym}\nabla'\gamma\, dA_g &= \mathrm{tr}_g \,\mathrm{Sym}\left({}^g\nabla + g\otimes\beta^\sharp - \beta\otimes\mathrm{Id} - \mathrm{Id}\otimes\beta + \alpha\right)\gamma\, dA_g\\ &= \mathrm{d}\star_g \gamma + \left(2\gamma(\beta^\sharp) - \gamma(\beta^\sharp) - \gamma(\beta^\sharp)\right) dA_g\\ &= \mathrm{d}\star_g \gamma,\end{aligned}$$

where we used that $\alpha(X)$ is trace-free and $[g]$-symmetric for all $X \in \Gamma(T\Sigma)$. Since the last summand of the right hand side of (3.4) is anti-symmetric, we obtain

$$\begin{aligned}\int_\Sigma \mathrm{tr}_g \,\mathrm{Ric}(\nabla)dA_g &= \int_\Sigma \mathrm{tr}_g \,\mathrm{Ric}(\nabla) + \int_\Sigma \mathrm{tr}_g \,\gamma^2 dA_g - \int_\Sigma \mathrm{tr}_g \,\mathrm{Sym}\nabla'\gamma\, dA_g\\ &= 4\pi\chi(\Sigma) - \|\alpha\|^2_g + \|\gamma\|^2_g - \int_\Sigma \mathrm{d}\star_g \gamma,\end{aligned}$$

thus the claim follows from the Stokes theorem. □

In [31] the following result is shown, albeit phrased in different language:

Proposition 3.4. *Let $(\Sigma, [g])$ be a Riemann surface. Then every torsion-free connection on $T\Sigma$ is projectively equivalent to a unique twisted $[g]$-Weyl connection.*

Let $\mathfrak{P}(\Sigma)$ denote the space of projective structures on Σ. Using Proposition 3.2 and Proposition 3.4 we immediately obtain:

Theorem 3.5. *Let $(\Sigma, [g])$ be a compact Riemann surface. Then*

$$\sup_{\mathfrak{p}\in\mathfrak{P}(\Sigma)} \inf_{\nabla\in\mathfrak{p}} \int_\Sigma \mathrm{tr}_g \,\mathrm{Ric}(\nabla)dA_g = 4\pi\chi(\Sigma).$$

Remark 3.6. A twisted Weyl connection ∇ on $(\Sigma, [g])$ defines an AH structure $(\mathfrak{p}(\nabla), [g])$ in the sense of [16, 17, 18], where $\mathfrak{p}(\nabla)$ denotes the projective equivalence class arising from ∇. Moreover, the twisted Weyl connection agrees with the *aligned representative* of the associated AH structure $(\mathfrak{p}(\nabla), [g])$. In particular, the equations (3.2) and (3.3) have counterparts in the equations (5.8) and (7.12) of [17]. Also, the Proposition 3.4 corresponds to the existence of a unique aligned representative for an AH structure from [17].

4. Submanifold theory of Lagrangian connections

We now restrict attention to torsion-free connections on $T\Sigma$ having symmetric Ricci tensor, so that the zero-section $o : \Sigma \to T^*\Sigma$ is a Lagrangian submanifold. If furthermore ∇ is timelike/spacelike, we obtain an induced metric $g = \mp o^* h_\nabla = \pm \mathrm{Ric}(\nabla)$ and the immersion $o : \Sigma \to T^*\Sigma$ has a well-defined normal bundle and second fundamental form. In particular, we want to compute when $o : \Sigma \to T^*\Sigma$ is *minimal*, that is, the trace with respect to g of its second fundamental form vanishes identically.

4.1. Algebraic preliminaries

Before we delve into the computations, we briefly review the relevant algebraic structure of the theory of oriented surfaces in an oriented Riemannian – and oriented split-signature Riemannian four manifold (M, g). We refer the reader to [4] and [11] for additional details.

First, let $X : \Sigma \to (M, g)$ be an immersion of an oriented surface Σ into an oriented Riemannian 4-manifold. The bundle of orientation compatible g-orthonormal coframes of (M, g) is an SO(4)-bundle $\pi : F_g^+ \to M$. The Grassmannian $G_2^+(\mathbb{R}^4)$ of oriented 2-planes in $\mathbb{R}^4$ is a homogeneous space for the natural action of SO(4) and the stabiliser subgroup is $\mathrm{SO}(2) \times \mathrm{SO}(2)$. Consequently, the pullback bundle $X^*F_g^+ \to \Sigma$ admits a reduction $F_X \subset X^*F_g^+$ with structure group $\mathrm{SO}(2) \times \mathrm{SO}(2)$, where the fibre of F_X at $p \in \Sigma$ consists of those coframes mapping the oriented tangent plane to Σ at $X(p)$ to some fixed oriented 2-plane in $\mathbb{R}^4$, while preserving the orientation.

The second fundamental form of X is a quadratic form on $T\Sigma$ with values in the rank two normal bundle of X. Therefore, it is represented by a map $F_X \to S^2(\mathbb{R}_2) \otimes \mathbb{R}^2$ which is equivariant with respect to some suitable representation of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ on $S^2(\mathbb{R}_2) \times \mathbb{R}^2$. The relevant representation is defined by the rule

$$\varrho(r_\alpha, r_\beta)(A)(x, y) = r_{-\beta} A(r_\alpha x, r_\alpha y), \quad x, y \in \mathbb{R}^2,$$

where $A \in S^2(\mathbb{R}_2) \otimes \mathbb{R}^2$ is a symmetric bilinear form on $\mathbb{R}^2$ with values in $\mathbb{R}^2$ and r_α, r_β denote counter-clockwise rotations in $\mathbb{R}^2$ by the angle α, β, respectively. As usual, we decompose the $\mathrm{SO}(2) \times \mathrm{SO}(2)$-module $S^2(\mathbb{R}_2) \otimes \mathbb{R}^2$ into irreducible pieces. This yields

$$\varrho = \varsigma_{0,-1} \oplus \varsigma_{2,1} \oplus \varsigma_{2,-1},$$

where for $(n, m) \in \mathbb{Z}^2$ the complex one-dimensional $\mathrm{SO}(2) \times \mathrm{SO}(2)$-representation $\varsigma_{n,m}$ is defined by the rule

$$\varsigma_{n,m}(r_\alpha, r_\beta) = \mathrm{e}^{\mathrm{i}(n\alpha + m\beta)}.$$

Explicitly, the relevant projections $S^2(\mathbb{R}_2)\otimes\mathbb{R}^2\to\mathbb{C}$ are

$$p_{0,-1}(A)=\frac{1}{2}\left(A^1_{11}+A^1_{22}\right)+\frac{\mathrm{i}}{2}\left(A^2_{11}+A^2_{22}\right),\tag{4.1}$$

$$p_{2,-1}(A)=\frac{1}{4}\left(A^1_{11}-A^1_{22}+2A^2_{12}\right)+\frac{\mathrm{i}}{4}\left(A^2_{11}-A^2_{22}-2A^1_{12}\right),\tag{4.2}$$

$$p_{2,1}(A)=\frac{1}{4}\left(A^1_{22}-A^1_{11}+2A^2_{12}\right)+\frac{\mathrm{i}}{4}\left(A^2_{11}-A^2_{22}+2A^1_{12}\right)\tag{4.3}$$

and where we write $A(e_i,e_j)=A^k_{ij}e_k$ with respect to the standard basis (e_1,e_2) of $\mathbb{R}^2$.

The canonical bundle K_Σ of Σ with respect to the complex structure induced by the metric X^*g and orientation is the bundle associated to the representation $\varsigma_{1,0}$. Moreover, the conormal bundle, thought of as a complex line bundle, is the bundle N^*_X associated to the representation $\varsigma_{0,1}$. Consequently, the second fundamental form of X defines a section H of the normal bundle which is the mean curvature vector of X, as well as a quadratic differential $Q_+$with values in the conormal bundle and a quadratic differential Q_- with values in the normal bundle. Consequently, we obtain a quartic differential Q_+Q_- on Σ which turns out to be holomorphic, provided X is minimal and g has constant sectional curvature, see [11].

If we instead consider a split-signature oriented Riemannian 4-manifold (M,g), the bundle of orientation compatible g-orthonormal coframes of (M,g) is an $\mathrm{SO}(2,2)$-bundle $\pi : F^+_g\to M$. Here, as usual, we take $\mathrm{SO}(2,2)$ to be the subgroup of $\mathrm{SL}(4,\mathbb{R})$ stabilising the quadratic form

$$q(x)=(x_1)^2+(x_2)^2-(y_1)^2-(y_2)^2,$$

where $(x,y)\in\mathbb{R}^{2,2}$. Now the action of $\mathrm{SO}(2,2)$ on the Grassmannian $G^+_2(\mathbb{R}^{2,2})$ of oriented 2-planes in $\mathbb{R}^{2,2}$ is not transitive, it is however transitive on the open submanifolds of oriented timelike/spacelike 2-planes. In both cases, the stabiliser subgroup is $\mathrm{SO}(2)\times\mathrm{SO}(2)$ as well. Therefore, the submanifold theory of a timelike/spacelike oriented surface in an oriented split-signature Riemannian four manifold is entirely analogous to the Riemannian case. In particular, we also encounter the mean curvature vector H and the quadratic differentials $Q_\pm$.

4.2. The mean curvature form

Knowing what to expect, we now carry out the submanifold theory of timelike/spacelike Lagrangian connections. Note however, that in addition to the split-signature metric h_∇, we also have a symplectic form Ω_∇. The symplectic form allows to identify the conormal bundle to a Lagrangian spacelike/timelike immersion with the cotangent bundle of Σ. In particular we may think of the mean curvature vector H as a 1-form, the conormal-bundle valued quadratic differential Q_+ as a cubic differential and the normal-bundle valued quadratic differential Q_- as a (1,0)-form.

The product $P := F\times\mathbb{R}_2$ is a principal right $\mathrm{GL}^+(2,\mathbb{R})$-bundle over $T^*\Sigma$, where the $\mathrm{GL}^+(2,\mathbb{R})$-right action is given by $(f,\xi)\cdot a=(a^{-1}\circ f,\xi a)$ for all

$a \in \mathrm{GL}^+(2,\mathbb{R})$ and $(f,\xi) \in P$. We define two $\mathbb{R}^2$-valued 1-forms on P

$$\rho := \frac{1}{2}\left(\psi^t + \omega\right) \quad \text{and} \quad \zeta := \frac{1}{2}\left(\psi^t - \omega\right),$$

so that the metric h_∇ satisfies

$$\pi^* h_\nabla = \rho^t\rho - \zeta^t\zeta = (\rho^1)^2 + (\rho^2)^2 - (\zeta^1)^2 - (\zeta^2)^2.$$

From the equivariance properties (2.7) and (2.8) of ψ and ω, we compute

$$R_a^*\begin{pmatrix}\rho\\ \zeta\end{pmatrix} = \frac{1}{2}\begin{pmatrix} a^t + a^{-1} & a^t - a^{-1} \\ a^t - a^{-1} & a^t + a^{-1}\end{pmatrix}\begin{pmatrix}\rho\\ \zeta\end{pmatrix}. \tag{4.4}$$

Remark 4.1. The reader may easily verify that the representation $\mathrm{GL}^+(2,\mathbb{R}) \to \mathrm{GL}(4,\mathbb{R})$ defined by (4.4) embeds $\mathrm{GL}^+(2,\mathbb{R})$ as a subgroup of $\mathrm{SO}(2,2)$.

Recall that the Lie algebra of the split-orthogonal group $\mathrm{O}(2,2)$ consists of matrices of the form

$$\begin{pmatrix}\mu & \nu \\ \nu^t & \vartheta\end{pmatrix}$$

where μ and ϑ are skew-symmetric. Consequently, there exist unique $\mathfrak{o}(2)$-valued 1-forms μ, ϑ on $F \times \mathbb{R}_2$ and a unique $\mathfrak{gl}(2,\mathbb{R})$-valued 1-form ν on $F \times \mathbb{R}_2$ such that

$$\mathrm{d}\begin{pmatrix}\rho\\ \zeta\end{pmatrix} = -\begin{pmatrix}\mu & \nu \\ \nu^t & \vartheta\end{pmatrix}\wedge\begin{pmatrix}\rho\\ \zeta\end{pmatrix}. \tag{4.5}$$

In order to compute these connection forms we first remark that since the function $S = (S_{ij})$ represents the (symmetric) Ricci tensor of ∇, there must exist unique real-valued functions $S_{ijk} = S_{jik}$ on F so that

$$\mathrm{d}S_{ij} = S_{ijk}\omega^k + S_{ik}\theta^k_j + S_{kj}\theta^k_i.$$

Clearly, the function $(S_{ijk}) : F \to S^2(\mathbb{R}_2)\otimes\mathbb{R}_2$ represents $\nabla\,\mathrm{Ric}(\nabla)$ with k being the derivative index.

Lemma 4.2. *We have*

$$\begin{aligned}
\mu_{ij} &= -\xi_{[i}\omega_{j]} + \theta_{[ij]} - S_{k[ij]}\omega^k,\\
\nu_{ij} &= -\xi_k\omega_k\delta_{ij} - \xi_{(i}\omega_{j)} - \theta_{(ij)} + S_{k[ij]}\omega_k,\\
\vartheta_{ij} &= -\xi_{[i}\omega_{j]} + \theta_{[ij]} + S_{k[ij]}\omega^k\omega_k,
\end{aligned}$$

where we write $\omega_i = \delta_{ij}\omega^j$ and $\theta_{ij} = \delta_{ik}\theta^k_j$.

Proof. Since the connection forms are unique, the proof amounts to plugging the above formulae into the structure equations (4.5) and verify that they are satisfied. This is tedious, but an elementary computation and hence is omitted. □

Recall that the components of ψ and ω – and hence equivalently the components of ρ and ζ – span the 1-forms on P that are semi-basic for the projection $\pi : P \to T^*\Sigma$. In particular, if ϵ is a 1-form on $T^*\Sigma$, then there exists a unique map $(e_1, e_2) : P \to \mathbb{R}_{2,2}$ so that $\pi^*\epsilon = e_1\rho + e_2\zeta$. Since $\pi^*\epsilon$ is invariant under the $\mathrm{GL}^+(2,\mathbb{R})$ right action, the function (e_1, e_2) satisfies the equivariance property determined by (4.4). Phrased differently, the cotangent bundle of $T^*\Sigma$ is the bundle associated to $\pi : P \to T^*\Sigma$ via the representation $\varrho : \mathrm{GL}^+(2,\mathbb{R}) \to \mathbb{R}_{2,2}$ defined by the rule

$$\varrho(a)\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} = \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}\frac{1}{2}\begin{pmatrix}a^t + a^{-1} & a^t - a^{-1}\\ a^t - a^{-1} & a^t + a^{-1}\end{pmatrix} \tag{4.6}$$

for all $a \in \mathrm{GL}^+(2,\mathbb{R})$ and (ξ_1, ξ_2) in $\mathbb{R}_{2,2}$.

We will next use this fact to exhibit the conormal bundle of the immersion $o : \Sigma \to T^*\Sigma$ as an associated bundle to a natural reduction of the pullback bundle o^*P. Note that by construction, the pullback bundle $o^*P \to \Sigma$ is just the frame bundle $\upsilon : F \to \Sigma$ and that on $o^*P \simeq F$ we have $\psi^t = -S\omega$, thus

$$\rho = \frac{1}{2}\left(\mathrm{I}_2 - S\right)\omega \quad \text{and} \quad \zeta = -\frac{1}{2}\left(\mathrm{I}_2 + S\right)\omega.$$

If we assume that ∇ is timelike/spacelike, then the Ricci tensor of ∇ is positive/negative definite and hence the equations $S = \pm\mathrm{I}_2$ define a reduction $F_\nabla \to \Sigma$ with structure group $\mathrm{SO}(2)$ whose basepoint projection we continue to denote by υ. Note that by construction, $F_\nabla \to \Sigma$ is the bundle of orientation preserving orthonormal coframes of the induced metric $g = \pm\mathrm{Ric}(\nabla)$. In particular, from (4.6) we see that the pullback bundle $o^*(T^*(T^*\Sigma))$ is the bundle associated to $\upsilon : F_\nabla \to \Sigma$ via the $\mathrm{SO}(2)$-representation on $\mathbb{R}_{2,2}$ defined by the rule

$$\varrho(a)\begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} = \begin{pmatrix}\xi_1 a^t & \xi_2 a^t\end{pmatrix} \tag{4.7}$$

for all $a \in \mathrm{SO}(2)$ and (ξ_1, ξ_2) in $\mathbb{R}_{2,2}$. Furthermore, on F_∇ we obtain $\psi = \mp\omega^t$ and hence

$$\begin{pmatrix}\rho\\ \zeta\end{pmatrix} = \begin{pmatrix}0\\ -\omega\end{pmatrix} \tag{4.8}$$

in the timelike case and

$$\begin{pmatrix}\rho\\ \zeta\end{pmatrix} = \begin{pmatrix}\omega\\ 0\end{pmatrix} \tag{4.9}$$

in the spacelike case. Recall that if α is a 1-form on Σ then there exists a unique $\mathbb{R}_2$-valued function a on F – and hence on F_∇ as well – so that $\upsilon^*\alpha = a\omega$. It follows as before that $T^*\Sigma$ is the bundle associated to F_∇ via the $\mathrm{SO}(2)$-representation defined by the rule

$$\varrho(a)(\xi) = \xi a^t \tag{4.10}$$

for all $a \in \mathrm{SO}(2)$ and $\xi \in \mathbb{R}_2$. Using (4.7), (4.8) and (4.9), we see that the conormal bundle

$$N^*_o := o^*(T^*(T^*\Sigma))/T^*\Sigma$$

of o is (isomorphic to) the bundle associated to F_∇ via the representation (4.10) as well. We thus have an isomorphism $N^*_o \simeq T^*\Sigma$ between the conormal bundle of the immersion $o : \Sigma \to T^*\Sigma$ and the cotangent bundle $T^*\Sigma$. Of course, the metric g on Σ provides an isomorphism $T^*\Sigma \simeq T\Sigma$ and hence $N_o \simeq T^*\Sigma$, where N_o denotes the normal bundle of o. The second fundamental form of o is a quadratic form on $T\Sigma$ with values in the normal bundle, thus here naturally a section of $S^2(T^*\Sigma) \otimes T^*\Sigma$.

Lemma 4.3. *Let $o : \Sigma \to T^*\Sigma$ be timelike/spacelike. Then the second fundamental form A of o is represented by the functions $A_{ijk} = A_{ikj}$, where*

$$A_{ijk} = \mp\frac{1}{2}\left(S_{kji} - S_{ijk} - S_{ikj}\right). \tag{4.11}$$

Proof. We will only treat the spacelike case, the timelike case is entirely analogous up to some sign changes. In our frame adaption on F_∇ we have $\zeta = 0$ and $\rho = \omega$. Consequently,

$$0 = \mathrm{d}\zeta = -\nu^t \wedge \rho - \vartheta \wedge \zeta = -\nu^t \wedge \omega$$

or in components

$$0 = \nu_{ji} \wedge \omega_j.$$

Cartan's lemma implies that there exist unique real-valued functions $A_{ijk} = A_{ikj}$ on F_∇ so that

$$\nu_{ji} = A_{ijk}\omega_k$$

and by standard submanifold theory the functions A_{ijk} represent the second fundamental form of o. In order to compute the functions A_{ijk}, we use that in our frame adaption $S_{ij} = -\delta_{ij}$ and hence

$$0 = \mathrm{d}S_{ij} = S_{ijk}\omega_k - \delta_{ik}\theta^k_j - \delta_{kj}\theta^k_i = S_{ijk}\omega_k - 2\theta_{(ij)}.$$

Since $\xi = 0$ we thus get from Lemma 4.2

$$\nu_{ji} = -\theta_{(ji)} + S_{k[ji]}\omega_k = \left(-\frac{1}{2}S_{ijk} + \frac{1}{2}S_{kji} - \frac{1}{2}S_{ikj}\right)\omega_k,$$

and the claim follows. □

Denoting by S^{ij} the functions on F representing the inverse of the Ricci curvature of ∇, so that $S^{ij}S_{jk} = \delta^i_k$, we thus have:

Theorem 4.4. *A timelike/spacelike Lagrangian connection ∇ is minimal if and only if*

$$S^{ij}\left(2S_{kij} - S_{ijk}\right) = 0. \tag{4.12}$$

Proof. By standard submanifold theory, the immersion $o : \Sigma \to T^*\Sigma$ is minimal if and only if the trace of the second fundamental form with respect to the induced metric $g = \pm\mathrm{Ric}(\nabla)$ vanishes identically. □

Remark 4.5. Note that in index notation the minimality condition (4.12) is equivalent to

$$\eta_k := \frac{1}{2}R^{ij}\left(2\nabla_i R_{jk} - \nabla_k R_{ij}\right) = 0,$$

where R_{ij} denotes the Ricci tensor of ∇ and R^{ij} its inverse. We call the 1-form η the *mean curvature form* of ∇.

Example 4.6. Let (Σ, g) be a two-dimensional Riemannian manifold. The Levi-Civita connection ∇ of g has Ricci tensor $\mathrm{Ric}(g) = Kg$, where K denotes the Gauss curvature of g. Thus, if K is positive/negative, then ∇ is a timelike/spacelike Lagrangian connection and Theorem 4.4 immediately implies that ∇ is minimal. In fact, we will show later (see Proposition 6.1) that on the 2-sphere metrics of positive Gauss curvature are the only examples of minimal Lagrangian connections.

Recall that a (non-degenerate) submanifold of a (pseudo-)Riemannian manifold is called *totally geodesic* if its second fundamental form vanishes identically. We call a timelike/spacelike connection ∇ *totally geodesic* if $o : \Sigma \to (T^*\Sigma, h_\nabla)$ is a totally geodesic submanifold. We also get:

Corollary 4.7. *Let ∇ be a timelike/spacelike Lagrangian connection. Then ∇ is totally geodesic if and only if its Ricci tensor is parallel with respect to ∇.*

Proof. The second fundamental form vanishes identically if and only if

$$S_{jki} = S_{ijk} - S_{ikj}.$$

Since the Ricci tensor is symmetric, the left hand side is symmetric in j, k, but the right hand side is anti-symmetric in j, k, thus S_{kji} vanishes identically. $\square$

4.3. Minimality and the Liouville curvature

The minimality condition for a timelike/spacelike Lagrangian connection can also be expressed in terms of the Liouville curvature of ∇. To this end we decompose the structure equation[3]

$$\mathrm{d}S_{ij} = S_{ijk}\omega^k + S_{ik}\theta^k_j + S_{kj}\theta^k_i \tag{4.13}$$

into (we compute modulo θ^i_j)

$$\begin{aligned}\mathrm{d}S_{ij} &= \left(\frac{1}{3}(S_{ijk} + S_{ikj} + S_{jki}) + \frac{2}{3}S_{ijk} - \frac{1}{3}(S_{ikj} + S_{jki})\right)\omega_k\\ &= \left(R_{ijk} + \frac{2}{3}L_{(i}\varepsilon_{j)k}\right)\omega^k,\end{aligned}$$

where we define

$$R_{ijk} = \frac{1}{3}\left(S_{ijk} + S_{ikj} + S_{jki}\right)$$

and

$$L_i = \frac{2}{3}\varepsilon^{jk}\left(S_{ijk} - \frac{1}{2}(S_{ikj} + S_{jki})\right).$$

Remark 4.8. The equivariance properties of the function $S = (S_{ij})$ yield $R_a^*L = La\det a$, where we write $L = (L_i)$. Since

$$R_a^*\left(\omega^1 \wedge \omega^2\right) = (\det a^{-1})\omega^1 \wedge \omega^2,$$

it follows that the there exists a unique 1-form $\lambda(\nabla)$ on Σ taking values in $\Lambda^2(T^*\Sigma)$, such that

$$\upsilon^*\lambda(\nabla) = \left(L_1\omega^1 + L_2\omega^2\right) \otimes \omega^1 \wedge \omega^2.$$

The $\Lambda^2(T^*\Sigma)$-valued 1-form was discovered by R. Liouville and hence we call it the Liouville curvature of ∇. Liouville showed that the vanishing of $\lambda(\nabla)$ is the complete obstruction to ∇ being projectively flat.

[3]We define $\varepsilon = (\varepsilon_{ij})$ by $\varepsilon_{ij} = -\varepsilon_{ji}$ with $\varepsilon_{12} = 1$ and ε^{ij} denote the components of the transpose inverse of ε.

Writing $g = \pm\mathrm{Ric}(\nabla)$ for the induced metric, we define $\beta \in \Omega^1(\Sigma)$ by

$$\beta = \frac{3}{8}\,\mathrm{tr}_g\,\mathrm{Sym}\nabla g,$$

where $\mathrm{Sym} : \Gamma\left(T^*\Sigma \otimes S^2(T^*\Sigma)\right) \to \Gamma\left(S^3(T^*\Sigma)\right)$ denotes the natural projection. We have:

Proposition 4.9. *A timelike/spacelike Lagrangian connection ∇ on $T\Sigma$ is minimal if and only if*

$$\lambda(\nabla) = \mp 2 \star_g \beta \otimes dA_g. \tag{4.14}$$

Proof. In order to prove the claim we work on the orthonormal coframe bundle of g which is cut out of the coframe bundle F by the equations $S_{ij} = \pm\delta_{ij}$. By definition, the functions S_{ijk} represent $\nabla\,\mathrm{Ric}(\nabla)$ and hence the functions R_{ijk} represent $\pm\mathrm{Sym}\nabla g$. Therefore, on F_g, writing $\upsilon^*\beta = b_i\omega_i$, the components b_i of β are

$$b_k = \pm\frac{3}{8}\delta^{ij}R_{ijk}. \tag{4.15}$$

Now on F_g the equation (4.14) becomes

$$(L_1\omega_1 + L_2\omega_2) \otimes \omega_1 \wedge \omega_2 = \mp 2\left(-b_2\omega_1 + b_1\omega_2\right) \otimes \omega_1 \wedge \omega_2$$

which is equivalent to

$$L_1 = \frac{3}{4}(R_{112} + R_{222}) \quad \text{and} \quad L_2 = -\frac{3}{4}\left(R_{111} + R_{221}\right),$$

where we have used that $\upsilon^*(\star_g\beta) = -b_2\omega_1 + b_1\omega_2$ as well as $\upsilon^* dA_g = \omega_1 \wedge \omega_2$ and (4.15). On the other hand Theorem 4.4 implies that the minimality is equivalent to

$$\begin{aligned}
\delta^{ij}\left(2S_{kij} - S_{ijk}\right) &= \delta^{ij}R_{ijk} + \delta^{ij}\left(\frac{4}{3}L_{(k}\varepsilon_{i)j} - \frac{2}{3}L_{(i}\varepsilon_{j)k}\right)\\
&= \delta^{ij}R_{ijk} + \delta^{ij}\left(\frac{2}{3}(L_k\varepsilon_{ij} + L_i\varepsilon_{kj}) - \frac{1}{3}(L_i\varepsilon_{jk} + L_j\varepsilon_{ik})\right)\\
&= \delta^{ij}R_{ijk} - \frac{4}{3}\delta^{ij}L_i\varepsilon_{jk} = 0.
\end{aligned}$$

Written out, this gives the two conditions

$$R_{111} + R_{221} + \frac{4}{3}L_2 = R_{112} + R_{222} - \frac{4}{3}L_1 = 0, \tag{4.16}$$

which proves the claim. □

Remark 4.10. Proposition 4.9 shows that a timelike/spacelike minimal Lagrangian connection is projectively flat if and only if the 1-form β vanishes identically.

5. Minimal Lagrangian connections

We will next compute the structure equations of a timelike/spacelike minimal Lagrangian connection ∇. As before, we work on the orthonormal coframe bundle F_g of the induced metric $g = \pm\mathrm{Ric}(\nabla)$. The submanifold theory discussed in §3 tells us that the second fundamental form of $o : \Sigma \to (T^*\Sigma, h_\nabla)$ is described in terms of a cubic differential and a (1,0)-form. Using the definition (4.2) of the (1,0)-form, the expression (4.11) for the second fundamental form and the minimality conditions (4.16), one easily computes that on F_g the (1,0)-form is represented by the complex-valued function

$$b = \pm\frac{3}{8}\left((R_{111} + R_{221}) - \mathrm{i}(R_{112} + R_{222})\right).$$

Hence comparing with (4.15), we conclude that $b = b_1 - \mathrm{i}b_2$. Note that if we define the (1,0)-form

$$\beta^{1,0} := \beta + \mathrm{i} \star_g \beta,$$

then we have $\upsilon^*\beta^{1,0} = (b_1 - \mathrm{i}b_2)(\omega_1 + \mathrm{i}\omega_2)$, thus the (1,0)-form obtained from the normal bundle valued quadratic differential Q_- by using the symplectic form Ω_∇ is $\beta^{1,0}$. Likewise, Q_+ gives the cubic differential C on Σ which is represented on F_g by the complex-valued function

$$c = \mp\frac{1}{8}\left((R_{111} - 3R_{122}) + \mathrm{i}\,(-3R_{112} + R_{222})\right).$$

From Lemma 2.1 and

$$\upsilon^*\left(\mathrm{Sym}_0\mathrm{Ric}(\nabla)\right) = \left(R_{ijk} - \frac{3}{2}\delta_{(ij}R_{k)lm}\delta^{lm}\right)\omega_i \otimes \omega_j \otimes \omega_k$$

we easily compute that the cubic differential C satisfies $\mathrm{Re}(C) = \mp\frac{1}{2}\mathrm{Sym}_0\nabla g$, where the subscript 0 denotes the trace-free part with respect to g.

The structure equations can now be summarised as follows:

Proposition 5.1. *Let Σ be an oriented surface and ∇ a timelike/spacelike minimal Lagrangian connection on $T\Sigma$. Then we obtain a triple (g, β, C) on Σ consisting of a Riemannian metric $g = \pm\mathrm{Ric}(\nabla)$, a 1-form $\beta = \frac{3}{8}\,\mathrm{tr}_g\,\mathrm{Sym}\nabla g$ and a cubic differential C so that $\mathrm{Re}(C) = \mp\frac{1}{2}\mathrm{Sym}_0\nabla g$. Furthermore, the triple (g, β, C) satisfies the following equations*

$$K_g = \pm 1 + 2\,|C|_g^2 + \delta_g\beta, \tag{5.1}$$

$$\bar\partial C = \left(\beta - \mathrm{i}\star_g\beta\right)\otimes C, \tag{5.2}$$

$$\mathrm{d}\beta = 0. \tag{5.3}$$

Proof. In our frame adaption where $S_{ij} = \pm\delta_{ij}$ on F_g, we obtain from (4.13)

$$0 = \mathrm{d}S_{ij} = \left(R_{ijk} + \frac{2}{3}L_{(i\varepsilon j)k}\right)\omega_k \pm \delta_{ik}\theta^k_j \pm \delta_{kj}\theta^k_i.$$

Therefore, writing $\theta_{ij} = \delta_{ik}\theta^k_j$, we have

$$\theta_{(ij)} = \mp\frac{1}{2}\left(R_{ijk} + \frac{2}{3}L_{(i\varepsilon j)k}\right)\omega_k. \tag{5.4}$$

For later usage we introduce the notation $c_1 = \mp(\frac{1}{8}R_{111} - \frac{3}{8}R_{122})$ and $c_2 = \mp(-\frac{3}{8}R_{112} + \frac{1}{8}R_{222})$, so that $c = c_1 + \mathrm{i}c_2$. Equation (5.4) written out gives

$$\begin{aligned} \theta_{11} &= \mp\frac{1}{2}R_{111}\omega_1 \mp \left(\frac{3}{4}R_{112} + \frac{1}{4}R_{222}\right)\omega_2, \\ \frac{1}{2}(\theta_{12}+\theta_{21}) &= \mp\left(\frac{3}{8}R_{112} - \frac{1}{8}R_{222}\right)\omega_1 \mp \left(\frac{3}{8}R_{122} - \frac{1}{8}R_{111}\right)\omega_2, \\ \theta_{22} &= \mp\left(\frac{3}{4}R_{122} + \frac{1}{4}R_{111}\right)\omega_1 \mp \frac{1}{2}R_{222}\omega_2. \end{aligned}$$

Defining

$$\varphi = \theta_{21} \mp \frac{1}{2}R_{222}\omega_1 \pm \left(\frac{1}{4}R_{111} + \frac{3}{4}R_{122}\right)\omega_2,$$

we compute

$$\theta = \begin{pmatrix} -\beta & \star_g\beta - \varphi \\ \varphi - \star_g\beta & -\beta \end{pmatrix} + \begin{pmatrix} c_1\omega^1 - c_2\omega^2 & -c_2\omega^1 - c_1\omega^2 \\ -c_2\omega^1 - c_1\omega^2 & -c_1\omega^1 + c_2\omega^2 \end{pmatrix}. \tag{5.5}$$

The motivation for the definition of φ is that we have

$$\mathrm{d}\omega_1 = -\omega_2 \wedge \varphi \quad \text{and} \quad \mathrm{d}\omega_2 = -\varphi \wedge \omega_1,$$

hence φ is the Levi-Civita connection form of g. In particular, we see that timelike/spacelike minimal Lagrangian connections are twisted Weyl connections. Since $\mathrm{Ric}(\nabla) = \pm g$, it follows that the curvature 2-form of θ must satisfy

$$\Theta = \mathrm{d}\theta + \theta\wedge\theta = \begin{pmatrix} 0 & \pm\omega_1\wedge\omega_2 \\ \mp\omega_1\wedge\omega_2 & 0 \end{pmatrix}. \tag{5.6}$$

In order to evaluate this condition we first recall that we write $\upsilon^*\beta = b_i\omega_i$ and

$$\begin{aligned} \mathrm{d}b_1 &= b_{11}\omega_1 + b_{12}\omega_2 + b_2\varphi, \\ \mathrm{d}b_2 &= b_{21}\omega_1 + b_{22}\omega_2 - b_1\varphi, \end{aligned}$$

for unique real-valued functions b_{ij} on F_g. From (5.5) and (5.6) we obtain

$$\mathrm{d}\beta = -\frac{1}{2}\left(\mathrm{d}\theta_{11} + \mathrm{d}\theta_{22}\right) = \frac{1}{2}\left(\theta_{12}\wedge\theta_{21} + \theta_{21}\wedge\theta_{12}\right) = 0$$

showing that β is closed, hence (5.3) is verified. Likewise, we also obtain

$$\begin{aligned} \mathrm{d}\varphi &= \frac{1}{2}(\mathrm{d}\theta_{21} - \mathrm{d}\theta_{12}) + \mathrm{d}\star_g\beta \\ &= (b_{11}+b_{22})\omega_1\wedge\omega_2 + \frac{1}{2}\left((\theta_{11}-\theta_{22})\wedge(\theta_{21}+\theta_{12})\right) \mp \omega_1\wedge\omega_2 \\ &= -\left(2\left((c_1)^2 + (c_2)^2\right) - (b_{11}+b_{22}) \pm 1\right)\omega_1\wedge\omega_2. \end{aligned}$$

Writing K_g for the Gauss curvature of g, this last equation is equivalent to

$$K_g = \pm 1 + 2\,|C|_g^2 + \delta_g\beta,$$

which verifies (5.1).

In order to prove (5.2), we use

$$\upsilon^*\left(\beta - \mathrm{i}\star_g\beta\right) = (b_1 + \mathrm{i}b_2)(\omega^1 - \mathrm{i}\omega^2).$$

In light of (2.6) the condition (5.2) is equivalent to the condition

$$\mathrm{d}c \wedge \omega = \overline{b}c\overline{\omega} \wedge \omega + 3\mathrm{i}c\varphi \wedge \omega, \tag{5.7}$$

where we use the complex notation $b = b_1 - \mathrm{i}b_2$, $c = c_1 + \mathrm{i}c_2$ and $\omega = \omega_1 + \mathrm{i}\omega_2$. Again, from (5.5) we compute

$$c\omega = \frac{1}{2}\left[(\theta_{11} - \theta_{22}) - \mathrm{i}\,(\theta_{12} + \theta_{21})\right],$$

hence

$$\begin{aligned}\mathrm{d}c \wedge \omega = \mathrm{d}(c\omega) - c\mathrm{d}\omega = -\theta_{12} \wedge \theta_{21}\\ + \frac{\mathrm{i}}{2}\left(\theta_{11} \wedge (\theta_{12} - \theta_{21}) + \theta_{22} \wedge (\theta_{21} - \theta_{12})\right) - (c_1 + \mathrm{i}c_2)(\mathrm{d}\omega_1 + \mathrm{i}\mathrm{d}\omega_2).\end{aligned}$$

Using (5.5) and the structure equations (2.5) this gives

$$\begin{aligned}\mathrm{d}c \wedge \omega = 3c_2\omega_1 \wedge \varphi + 3c_1\omega_2 \wedge \varphi - 2(b_1c_2 + b_2c_1)\omega_1 \wedge \omega_2\\ + \mathrm{i}\left(-3c_1\omega_1 \wedge \varphi + 3c_2\omega_2 \wedge \varphi + 2(b_1c_1 - b_2c_2)\omega_1 \wedge \omega_2\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\mathrm{d}c \wedge \omega = (b_1 + \mathrm{i}b_2)(c_1 + \mathrm{i}c_2)(\omega_1 - \mathrm{i}\omega_2) \wedge (\omega_1 + \mathrm{i}\omega_2)\\ + 3\mathrm{i}(c_1 + \mathrm{i}c_2)\varphi \wedge (\omega_1 + \mathrm{i}\omega_2),\end{aligned}$$

that is, equation (5.7). This completes the proof. □

Conversely, unravelling our computations backwards, we also get:

Proposition 5.2. *Suppose a triple (g, β, C) on an oriented surface Σ satisfies the equations (5.1),(5.2),(5.3). Then the connection form (5.5) on F_g defines a timelike/spacelike minimal Lagrangian connection ∇ on $T\Sigma$ with* $\mathrm{Ric}(\nabla) = \pm g$.

We immediately obtain:

Corollary 5.3. *Let Σ be an oriented surface. Then there exists a one-to-one correspondence between timelike/spacelike minimal Lagrangian connections on $T\Sigma$ and triples (g, β, C) satisfying (5.1),(5.2),(5.3).*

Proof. Clearly, the map sending a torsion-free minimal Lagrangian connection ∇ into the set of triples (g, β, C) satisfying the above structure equations, is surjective. Now suppose the two triples (g_1, β_1, C_1) and (g_2, β_2, C_2) on Σ satisfy the above structure equations and define the same torsion-free spacelike minimal Lagrangian connection ∇ on $T\Sigma$. Then $g_1 = \pm\mathrm{Ric}(\nabla) = g_2$ and consequently we obtain $\beta_1 = \beta_2$ as well as $C_1 = C_2$, since these quantities are defined in terms of $\nabla\mathrm{Ric}(\nabla)$ by using the metric $g_1 = g_2$. □

Remark 5.4. Remark 4.10 immediately implies that a minimal Lagrangian connection is projectively flat if and only if the cubic differential C is holomorphic.

Another consequence of the structure equation is:

Proposition 5.5. *Let ∇ be a timelike/spacelike Lagrangian connection that is totally geodesic. Then ∇ is the Levi-Civita connection of a metric g of Gauss curvature $K_g = \pm 1$.*

Proof. The Lagrangian connection ∇ is totally geodesic if and only if the second fundamental form vanishes identically or equivalently, if β and C vanish identically. In this case (5.5) implies that θ is the Levi-Civita connection of g and the structure equation (5.1) gives that g has Gauss curvature ± 1. □

Remark 5.6. As we have mentioned previously in Remark 3.6, a twisted Weyl connection on $(\Sigma, [g])$ defines an AH structure $(\mathfrak{p}(\nabla), [g])$. Moreover, a twisted Weyl connection arising from a triple (g, β, C) satisfying $\overline{\partial} C = (\beta - \mathrm{i} \star_g \beta) \otimes C$ defines an associated AH structure $(\mathfrak{p}(\nabla), [g])$ which is *naive Einstein* in the terminology of [16, 17, 18]. Therefore, every minimal Lagrangian connection defines a naive Einstein AH structure.

6. The spherical case

The system of equations governing minimal Lagrangian connections are easy to analyse on the 2-sphere S^2:

Proposition 6.1. *A connection on the tangent bundle of S^2 is minimal Lagrangian if and only if it is the Levi-Civita connection of a metric of positive Gauss curvature.*

Proof. Let ∇ be a minimal Lagrangian connection on TS^2 with associated triple (g, β, C). Since β is closed and $H^1(S^2) = 0$, the 1-form β is exact and hence there exists a smooth real-valued function r on S^2 such that $\beta = \mathrm{d}r$. Hence we have

$$\overline{\partial} C = \big(\mathrm{d}r - \mathrm{i} \star_g \mathrm{d}r\big) \otimes C.$$

Observe that $\mathrm{d}r - \mathrm{i} \star_g \mathrm{d}r = 2\overline{\partial} r$, therefore, the cubic differential $\mathrm{e}^{-2r} C$ is holomorphic. Since, by Riemann-Roch, there are no non-trivial cubic holomorphic differentials on the 2-sphere, C must vanish identically. The connection form (5.5) of ∇ thus becomes

$$\theta = \begin{pmatrix} -\mathrm{d}r & \star_g \mathrm{d}r - \varphi \\ \varphi - \star_g \mathrm{d}r & -\mathrm{d}r \end{pmatrix},$$

where φ denotes the Levi-Civita connection form of g. We conclude that ∇ is a Weyl connection given by

$$\nabla = {}^g\nabla + g \otimes {}^g\nabla r - \mathrm{d}r \otimes \mathrm{Id} - \mathrm{Id} \otimes \mathrm{d}r,$$

where ${}^g\nabla r$ denotes the gradient of r with respect to g. Since the Levi-Civita connection of a Riemannian metric g transforms under conformal change as [2, Theorem 1.159]

$$^{\exp(2f)g}\nabla = {}^g\nabla - g \otimes {}^g\nabla f + \mathrm{d}f \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{d}f,$$

we obtain $\nabla = {}^{\exp(-2r)g}\nabla$, thus showing that ∇ is the Levi-Civita connection of a Riemannian metric. Moreover, since $\mathrm{Ric}(\nabla)$ must be positive or negative definite, the Gauss curvature of the metric $\mathrm{e}^{-2r} g$ cannot vanish and hence is positive by the Gauss–Bonnet theorem. Finally, Example 4.6 shows that conversely the Levi-Civita connection of a Riemannian metric of positive Gauss curvature defines a minimal Lagrangian connection, thus completing the proof. □

7. The case of negative Euler-characteristic

Before we address the classification of minimal Lagrangian connections on compact surfaces of negative Euler characteristic, we observe that every projectively flat spacelike minimal Lagrangian connection defines a properly convex projective structure. Indeed, Labourie gave the following characterisation of properly convex projective manifolds:

Theorem 7.1 (Labourie [26], Theorem 3.2.1)**.** *Let $(M, \mathfrak{p})$ be an oriented flat projective manifold. Then the following statements are equivalent:*

(i) $\mathfrak{p}$ *is properly convex;*

(ii) *there exists a connection $\nabla \in \mathfrak{p}$ preserving a volume form and whose Ricci curvature is negative definite.*

We immediately obtain:

Corollary 7.2. *Let ∇ be a projectively flat spacelike minimal Lagrangian connection on the oriented surface Σ. Then ∇ defines a properly convex projective structure.*

Proof. In Remark 4.10 we have seen that a minimal Lagrangian connection ∇ is projectively flat if and only if β vanishes identically. In the projectively flat case the connection 1-form θ of ∇ thus is (see (5.5))

$$\theta = \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} + \begin{pmatrix} c_1\omega^1 - c_2\omega^2 & -c_2\omega^1 - c_1\omega^2 \\ -c_2\omega^1 - c_1\omega^2 & -c_1\omega^1 + c_2\omega^2 \end{pmatrix}.$$

In particular, the trace of θ vanishes identically and hence ∇ preserves the volume form of g. Since $\mathrm{Ric}(\nabla) = -g$, the claim follows by applying Labourie's result. $\square$

7.1. Classification

In Section 5 we have seen that a triple (g, β, C) on an oriented surface Σ satisfying (5.1),(5.2),(5.3) uniquely determines a minimal Lagrangian connection on $T\Sigma$. In this section we will show that in the case where Σ is compact and has negative Euler characteristic $\chi(\Sigma)$, the conformal equivalence $[g]$ of g and the cubic differential C also uniquely determine (g, β, C) and hence the connection, provided C does not vanish identically. In the case where C does vanish identically the connection is determined uniquely in terms of $[g]$ and β.

We start by showing that there are no timelike minimal Lagrangian connections on a compact oriented surface of negative Euler-characteristic (the reader may also compare this with [17, Theorem 5.4]).

Proposition 7.3. *Suppose ∇' is a minimal Lagrangian connection on the compact oriented surface Σ satisfying $\chi(\Sigma) < 0$. Then ∇' is spacelike.*

Proof. Suppose ∇' were timelike and let $g = \mathrm{Ric}(\nabla')$. Then we obtain

$$\int_\Sigma \mathrm{tr}_g \, \mathrm{Ric}(\nabla') dA_g = 2 \int_\Sigma dA_g = 2 \, \mathrm{Area}(\Sigma, g) \geqslant 0$$

and hence Proposition 3.2 and Theorem 3.5 imply that

$$4\pi\chi(\Sigma) = \sup_{\mathfrak{p} \in \mathfrak{P}(\Sigma)} \inf_{\nabla \in \mathfrak{p}} \int_\Sigma \mathrm{tr}_g \, \mathrm{Ric}(\nabla) dA_g \geqslant 0,$$

a contradiction. □

Without losing generality we henceforth assume that the torsion-free minimal Lagrangian connection ∇ on a compact oriented surface Σ with $\chi(\Sigma) < 0$ is spacelike. We will show that the triple (g, β, C) defined by ∇ is uniquely determined in terms of $[g]$ and (β, C).

Suppose (g, β, C) with β closed satisfy

$$K_g = -1 + 2\,|C|_g^2 + \delta_g \beta.$$

Let g_0 denote the hyperbolic metric in $[g]$ and write $g = \mathrm{e}^{2u} g_0$, so that

$$\mathrm{e}^{-2u}(-1 - \Delta_{g_0} u) = -1 + 2\mathrm{e}^{-6u}|C|_{g_0}^2 + \mathrm{e}^{-2u}\delta_{g_0}\beta.$$

We obtain

$$-\Delta_{g_0} u = 1 + \delta_{g_0}\beta - \mathrm{e}^{2u} + 2\mathrm{e}^{-4u}|C|_{g_0}^2.$$

Omitting henceforth reference to g_0 we will show:

Theorem 7.4. *Let (Σ, g_0) be a compact hyperbolic Riemann surface. Suppose $\beta \in \Omega^1(\Sigma)$ is closed and C is a cubic differential on Σ. Then the equation*

$$-\Delta u = 1 + \delta\beta - \mathrm{e}^{2u} + 2\mathrm{e}^{-4u}|C|^2 \tag{7.1}$$

admits a unique solution $u \in C^\infty(\Sigma)$.

Using the Hodge decomposition theorem it follows from the closedness of β that we may write $\beta = \gamma + \mathrm{d}v$ for a real-valued function $v \in C^\infty(\Sigma)$ and a unique harmonic 1-form $\gamma \in \Omega^1(\Sigma)$. Since γ is harmonic, it is co-closed, hence (7.1) becomes

$$\Delta u = -1 - \delta\mathrm{d}v + \mathrm{e}^{2u} - 2\mathrm{e}^{-4u}|C|^2 = -1 + \Delta v + \mathrm{e}^{2u} - 2\mathrm{e}^{-4u}|C|^2.$$

Writing $u' := u - v$, we obtain

$$\Delta u' = -1 + e^{2(u'+v)} - 2\mathrm{e}^{-4(u'+v)}|C|^2.$$

Using the notation $\kappa = -\mathrm{e}^{2v} < 0$ and $\tau = \mathrm{e}^{-4v}|C|^2$, as well as renaming $u := u'$, we see that (7.4) follows from:

Theorem 7.5. *Let (Σ, g_0) be a compact hyperbolic Riemann surface. Suppose $\kappa, \tau \in C^\infty(\Sigma)$ satisfy $\kappa < 0$ and $\tau \geqslant 0$. Then the equation*

$$-\Delta u = 1 + \kappa\mathrm{e}^{2u} + 2\tau\mathrm{e}^{-4u} \tag{7.2}$$

admits a unique solution $u \in C^\infty(\Sigma)$.

Remark 7.6. This theorem can also be proved using the technique of sub – and supersolutions, see [17, Chapter 9]. Here we instead use techniques from the calculus of variations.

In order to prove this theorem we define an appropriate functional $\mathcal{E}_{\kappa,\tau}$ on the Sobolev space $W^{1,2}(\Sigma)$. As usual, we say a function $u \in W^{1,2}(\Sigma)$ is a *weak solution* of (7.2) if for all $\phi \in C^\infty(\Sigma)$

$$0 = \int_\Sigma -\langle \mathrm{d}u, \mathrm{d}\phi\rangle + \left(1 + \kappa\mathrm{e}^{2u} + 2\tau\mathrm{e}^{-4u}\right)\phi\, dA. \tag{7.3}$$

Note that this definition makes sense. Indeed, it follows from the Moser–Trudinger inequality that the exponential map sends the Sobolev space $W^{1,2}(\Sigma)$ into $L^p(\Sigma)$ for every $p < \infty$, hence the right hand side of (7.3) is well defined.

Lemma 7.7. *Suppose $u \in W^{1,2}(\Sigma)$ is a critical point of the functional*

$$\mathcal{E}_{\kappa,\tau} : W^{1,2}(\Sigma) \to \overline{\mathbb{R}}, \quad u \mapsto \frac{1}{2}\int_\Sigma |\mathrm{d}u|^2 - 2u - \kappa \mathrm{e}^{2u} + \tau \mathrm{e}^{-4u} dA.$$

Then $u \in C^\infty(\Sigma)$ and u solves (7.2).

Proof. For $u, v \in W^{1,2}(\Sigma)$ we define $\gamma_{u,v}(t) = u + tv$ for $t \in \mathbb{R}$. We consider the curve $\Gamma_{u,v} = \mathcal{E}_{\kappa,\tau} \circ \gamma_{u,v} : \mathbb{R} \to \mathbb{R}$ so that

$$\begin{aligned}\Gamma_{u,v}(t) = \frac{1}{2}\int_\Sigma & |\mathrm{d}u|^2 + 2t\langle \mathrm{d}u, \mathrm{d}v\rangle + t^2|\mathrm{d}v|^2 \\ & -2(u+tv) - \kappa \mathrm{e}^{2(u+tv)} + \tau \mathrm{e}^{-4(u+tv)} dA.\end{aligned}$$

The curve $\Gamma_{u,v}(t)$ is differentiable in t with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{u,v}(t) = \int_\Sigma \langle \mathrm{d}u, \mathrm{d}v\rangle + t|\mathrm{d}v|^2 - v - v\kappa \mathrm{e}^{2(u+tv)} - 2v\tau \mathrm{e}^{-4(u+tv)} dA.$$

Note that this last expression is well-defined. Again, it follows from the Moser–Trudinger inequality that $\mathrm{e}^{2(u+tv)} \in L^2(\Sigma)$ for all $u, v \in W^{1,2}(\Sigma)$ and $t \in \mathbb{R}$. Since $W^{1,2}(\Sigma) \subset L^2(\Sigma)$ it follows that $v\mathrm{e}^{2(u+tv)}$ is in $L^1(\Sigma)$ by Hölder's inequality and thus so is $v\mathrm{e}^{-4(u+tv)}$. In particular, assuming that u is a critical point and setting $t = 0$ after differentiation gives

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Gamma_{u,v}(t) = \int_\Sigma \langle \mathrm{d}u, \mathrm{d}v\rangle - v - v\kappa \mathrm{e}^{2u} - 2v\tau \mathrm{e}^{-4u} dA.$$

Since $C^\infty(\Sigma) \subset W^{1,2}(\Sigma)$ it follows that u is a weak solution of (7.2). Since the right hand side of (7.2) is in $L^p(\Sigma)$ for all $p < \infty$, it follows from the Caldéron-Zygmund inequality that $u \in W^{2,p}(\Sigma)$ for any $p < \infty$. Therefore, by the Sobolev embedding theorem, u is an element of the Hölder space $C^{1,\alpha}(\Sigma)$ for any $\alpha < 1$. Since the right hand side of (7.2) is Hölder continuous in u, it follows from Schauder theory that $u \in C^2(\Sigma)$, so that u is a classical solution of (7.2). Iteration of the Schauder estimates then gives that $u \in C^\infty(\Sigma)$. □

Since $\tau \geqslant 0$ we have $\mathcal{E}_{\kappa,\tau} \geqslant \mathcal{E}_{\kappa,0}$ where here 0 stands for the zero-function. The functional $\mathcal{E}_{\kappa,0}$ appears in the variational formulation of the equation for prescribed Gauss curvature κ of a metric $g = \mathrm{e}^{2u}g_0$ on Σ. In particular, $\mathcal{E}_{\kappa,0}$ is well-known to be coercive and hence so is $\mathcal{E}_{\kappa,\tau}$. In addition, we have:

Lemma 7.8. *The functional $\mathcal{E}_{\kappa,\tau}$ is strictly convex on $W^{1,2}(\Sigma)$.*

Proof. Let $u, v \in W^{1,2}(\Sigma)$ be given. Using the notation of the previous lemma, we observe that $\Gamma_{u,v}(t)$ is twice differentiable in t with derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Gamma_{u,v}(t) = \int_\Sigma |\mathrm{d}v|^2 - 2v^2\kappa \mathrm{e}^{2(u+tv)} + 8v^2\tau \mathrm{e}^{-4(u+tv)} dA. \tag{7.4}$$

Note again that by Sobolev embedding $v^2 \in L^2(\Sigma)$ for $v \in W^{1,2}(\Sigma)$ and that both $\mathrm{e}^{2(u+tv)}$ and $\mathrm{e}^{-4(u+tv)}$ are in $L^2(\Sigma)$, hence the right hand side of the equation

(7.4) is well-defined by Hölder's inequality. In particular, computing the second variation gives

$$\begin{aligned}\mathcal{E}''_{\kappa,\tau}(u)[v,v] &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\mathcal{E}_{\kappa,\tau}(u+tv)\\ &= \int_\Sigma |\mathrm{d}v|^2 dA + 2\int_\Sigma v^2(4\tau - \mathrm{e}^{6u}\kappa)\mathrm{e}^{-4u} dA\\ &\geqslant \|\mathrm{d}v\|^2_{L^2(\Sigma)},\end{aligned}$$

where we have used that $\tau \geqslant 0$ and $\kappa < 0$. Since for a non-zero constant function v we obviously have $\mathcal{E}''_{\kappa,\tau}(u)[v,v] > 0$ it follows that the quadratic form $\mathcal{E}''_{\kappa,\tau}$ is positive definite on $W^{1,2}(\Sigma)$. Hence, the claim is proved. □

Proof of Theorem 7.5. We have shown that $\mathcal{E}_{\kappa,\tau}$ is a continuous strictly convex coercive functional on the reflexive Banach space $W^{1,2}(\Sigma)$, hence $\mathcal{E}_{\kappa,\tau}$ attains a unique minimum on $W^{1,2}(\Sigma)$, see for instance [34]. Since we know that the minimum is smooth, Theorem 7.5 is proved. □

We define the area of a timelike/spacelike connection to be the area of $o(\Sigma) \subset (T^*\Sigma, h_\nabla)$. We have:

Theorem 7.9. *Let ∇ be a minimal Lagrangian connection on the compact oriented surface Σ with $\chi(\Sigma) < 0$. Then we have*

$$\mathrm{Area}(\nabla) = -2\pi\chi(\Sigma) + 2\|C\|^2_g.$$

Proof. We have seen that the Gauss curvature of the metric $o^*h_\nabla = g = -\mathrm{Ric}(\nabla)$ defined by a minimal Lagrangian connection ∇ on Σ satisfies

$$K_g = -1 + 2|C|^2_g + \delta_g\beta.$$

Integrating against dA_g and using the Stokes and Gauss–Bonnet theorem gives

$$2\pi\chi(\Sigma) = -\mathrm{Area}(\nabla) + 2\|C\|^2_g,$$

thus proving the claim. □

Remark 7.10. An obvious consequence of Theorem 7.9 is the area inequality

$$\mathrm{Area}(\nabla) \geqslant -2\pi\chi(\Sigma) \tag{7.5}$$

holding for minimal Lagrangian connections. Recall that if ∇ is projectively flat, then the projective structure defined by ∇ is properly convex. Labourie [26] associated to every properly convex projective surface $(\Sigma, \mathfrak{p})$ a unique minimal mapping from the universal cover $\tilde{\Sigma}$ to the symmetric space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ which satisfies the very same area inequality, that is (7.5), see [27]. Moreover, it is shown in [27] that equality holds if and only if $\mathfrak{p}$ is defined by the Levi-Civita connection of a hyperbolic metric.

Definition 7.11. We call a minimal Lagrangian connection ∇ *area minimising* if ∇ has area $-2\pi\chi(\Sigma)$.

Remark 7.12. Theorem 7.9 shows that a minimal Lagrangian connection ∇ is area minimising if and only if the induced cubic differential vanishes identically. We have shown in Proposition 5.5 that in the projectively flat case – when β vanishes identically – this translates to ∇ being the Levi-Civita connection of a hyperbolic metric, a statement in agreement with [27].

Theorem 7.4 shows that the triple (g, β, C) is uniquely determined in terms of the conformal equivalence class $[g]$, the cubic differential C and the 1-form β on Σ. Since C can locally be rescaled to be holomorphic, its zeros must be isolated and hence β is uniquely determined by C provided C does not vanish identically. Therefore, applying Corollary 5.3 shows:

Theorem A. *Let Σ be a compact oriented surface with $\chi(\Sigma) < 0$. Then we have:*

(i) *there exists a one-to-one correspondence between area minimising Lagrangian connections on $T\Sigma$ and pairs $([g], \beta)$ consisting of a conformal structure $[g]$ and a closed 1-form β on Σ;*

(ii) *there exists a one-to-one correspondence between non-area minimising minimal Lagrangian connections on $T\Sigma$ and pairs $([g], C)$ consisting of a conformal structure $[g]$ and a non-trivial cubic differential C on Σ that satisfies $\overline{\partial} C = \big(\beta - \mathrm{i} \star_g \beta\big) \otimes C$ for some closed 1-form β.*

7.2. Concluding remarks

Remark 7.13. We have proved that on a compact oriented surface Σ of negative Euler characteristic we have a bijective correspondence between projectively flat spacelike minimal Lagrangian connections and pairs $([g], C)$ consisting of a conformal structure $[g]$ and a holomorphic cubic differential C on Σ. By the work of Labourie [26] and Loftin [30], the latter set is also in bijective correspondence with the properly convex projective structures on Σ. Since by Corollary 7.2 every projectively flat spacelike minimal Lagrangian connection defines a properly convex projective structure, we conclude that every such projective structure arises from a unique projectively flat spacelike minimal Lagrangian connection.

Remark 7.14. Recall that the universal cover $\tilde{\Sigma}$ of a properly convex projective surface $(\Sigma, \mathfrak{p})$ is a convex subset of $\mathbb{RP}^2$. Pulling back the minimal Lagrangian connection $\nabla \in \mathfrak{p}$ to the universal cover gives a section of $A \to \tilde{\Sigma}$, where now, by the work of Libermann [28], the total space of the affine bundle $A \to \tilde{\Sigma}$ is contained in the submanifold of the para-Kähler manifold $A_0 \subset \mathbb{RP}^2 \otimes \mathbb{RP}^{2*}$ consisting of non-incident point-line pairs. In particular, we obtain a minimal Lagrangian immersion $\tilde{\Sigma} \to A$, recovering the result of Hildebrand [22, 23] in the case of two dimensions. Therefore, using the result of Loftin [30], one should be able to show that every properly convex projective manifold arises from a minimal Lagrangian connection.

Remark 7.15. The case of the 2-torus can be treated with similar techniques, except for the possible occurrence of Lorentzian minimal Lagrangian connections. This will be addressed elsewhere.

Remark 7.16. In higher dimensions, the class of totally geodesic Lagrangian connections contains the Levi-Civita connection of Einstein metrics of non-zero scalar curvature. We also refer the reader to [7, 21] for a study of Einstein metrics in projective geometry.

Remark 7.17. In [31], the author has introduced the notion of an extremal conformal structure for a projective manifold $(M, \mathfrak{p})$. In two-dimensions, the naive Einstein AH structures of [17] appear to provide examples of projective surfaces admitting an extremal conformal structure. This may be taken up in future work.

References

[1] Y. BENOIST, D. HULIN, Cubic differentials and finite volume convex projective surfaces, *Geom. Topol.* **17** (2013), 595–620. DOI 10.2140/gt.2013.17.595 MR 3039771 zbM 1266.30030 2

[2] A. L. BESSE, *Einstein manifolds*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* **10**, Springer-Verlag, Berlin, 1987. DOI 10.1007/978-3-540-74311-8 MR 867684 zbM 0613.53001 23

[3] S. B. BRADLOW, Vortices in holomorphic line bundles over closed Kähler manifolds, *Comm. Math. Phys.* **135** (1990), 1–17. MR 1086749 zbM 0717.53075 3

[4] R. L. BRYANT, Conformal and minimal immersions of compact surfaces into the 4-sphere, *J. Differential Geom.* **17** (1982), 455–473. MR 679067 zbM 0498.53046 13

[5] D. M. J. CALDERBANK, Möbius structures and two-dimensional Einstein-Weyl geometry, *J. Reine Angew. Math.* **504** (1998), 37–53. DOI 10.1515/crll.1998.111 MR 1656822 zbM 0909.53029 4

[6] D. M. J. CALDERBANK, Selfdual 4-manifolds, projective surfaces, and the Dunajski-West construction, *SIGMA Symmetry Integrability Geom. Methods Appl.* **10** (2014), Paper 035, 18. DOI 10.3842/SIGMA.2014.035 MR 3210600 zbM 1288.53075 9

[7] A. ČAP, A. R. GOVER, H. R. MACBETH, Einstein metrics in projective geometry, *Geom. Dedicata* **168** (2014), 235–244. DOI 10.1007/s10711-013-9828-3 MR 3158041 zbM 1295.53011 28

[8] A. ČAP, T. METTLER, Geometric theory of Weyl structures, *Commun. Contemp. Math. (to appear)* (2022). arXiv:1908.10325 5

[9] S. Y. CHENG, S. T. YAU, On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial s x_j) = F(x,u)$, *Comm. Pure Appl. Math.* **30** (1977), 41–68. DOI 10.1002/cpa.3160300104 MR 0437805 zbM 0347.35019 1

[10] S. Y. CHENG, S.-T. YAU, Complete affine hypersurfaces. I. The completeness of affine metrics, *Comm. Pure Appl. Math.* **39** (1986), 839–866. DOI 10.1002/cpa.3160390606 MR 859275 zbM 0623.53002 1

[11] S. S. CHERN, J. G. WOLFSON, Minimal surfaces by moving frames, *Amer. J. Math.* **105** (1983), 59–83. DOI 10.2307/2374381 MR 692106 zbM 0521.53050 13, 14

[12] S. CHOI, W. M. GOLDMAN, Convex real projective structures on closed surfaces are closed, *Proc. Amer. Math. Soc.* **118** (1993), 657–661. DOI 10.2307/2160352 MR 1145415 zbM 0810.57005 1

[13] K. CIELIEBAK, A. R. GAIO, I. MUNDET I RIERA, D. A. SALAMON, The symplectic vortex equations and invariants of Hamiltonian group actions, *J. Symplectic Geom.* **1** (2002), 543–645. MR 1959059 3

[14] D. DUMAS, M. WOLF, Polynomial cubic differentials and convex polygons in the projective plane, *Geom. Funct. Anal.* **25** (2015), 1734–1798. DOI 10.1007/s00039-015-0344-5 MR 3432157 zbM 06526259 2, 4

[15] M. DUNAJSKI, T. METTLER, Gauge theory on projective surfaces and anti-self-dual Einstein metrics in dimension four, *J. Geom. Anal.* **28** (2018), 2780–2811. DOI 10.1007/s12220-017-9934-9 MR 3833818 zbM 07002652 2, 9

[16] D. J. F. FOX, Geometric structures modeled on affine hypersurfaces and generalizations of the Einstein–Weyl and affine hypersphere equations, 2009. arXiv:0909.1897 4, 13, 23

[17] D. J. F. FOX, Einstein-like geometric structures on surfaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **12** (2013), 499–585. MR 3137456 zbM 1285.53006 2, 4, 13, 23, 24, 25, 29

[18] D. J. F. FOX, Geometric structures modeled on affine hypersurfaces and generalizations of the Einstein–Weyl and affine sphere equations, in *Extended abstracts Fall 2013—geometrical analysis, type theory, homotopy theory and univalent foundations*, *Trends Math. Res. Perspect. CRM Barc.* **3**, Birkhäuser/Springer, Cham, 2015, pp. 15–19. MR 3738964 4, 13, 23

[19] D. J. F. Fox, A Schwarz lemma for Kähler affine metrics and the canonical potential of a proper convex cone, *Ann. Mat. Pura Appl. (4)* **194** (2015), 1–42. DOI 10.1007/s10231-013-0362-6 MR 3303003 2

[20] W. M. Goldman, Convex real projective structures on compact surfaces, *J. Differential Geom.* **31** (1990), 791–845. MR 1053346 zbM 0711.53033 1

[21] A. R. Gover, H. R. Macbeth, Detecting Einstein geodesics: Einstein metrics in projective and conformal geometry, *Differential Geom. Appl.* **33** (2014), 44–69. DOI 10.1016/j.difgeo.2013.10.011 MR 3159950 zbM 1372.53017 28

[22] R. Hildebrand, The cross-ratio manifold: a model of centro-affine geometry, *Int. Electron. J. Geom.* **4** (2011), 32–62. MR 2854275 zbM 1308.53023 2, 28

[23] R. Hildebrand, Half-dimensional immersions in para-Kähler manifolds, *Int. Electron. J. Geom.* **4** (2011), 85–113. MR 2854277 zbM 1308.53093 2, 28

[24] N. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc. (3)* **55** (1987), 59–126. DOI 10.1112/plms/s3-55.1.59 MR 887284 zbM 0634.53045 3

[25] N. Hitchin, Lie groups and Teichmüller space, *Topology* **31** (1992), 449–473. DOI 10.1016/0040-9383(92)90044-I MR 1174252 zbM 0769.32008 1

[26] F. Labourie, Flat projective structures on surfaces and cubic holomorphic differentials, *Pure Appl. Math. Q.* **3** (2007), 1057–1099. MR 2402597 zbM 1158.32006 2, 3, 4, 24, 27, 28

[27] F. Labourie, Cyclic surfaces and Hitchin components in rank 2, *Ann. of Math. (2)* **185** (2017), 1–58. DOI 10.4007/annals.2017.185.1.1 MR 3583351 zbM 06686583 27, 28

[28] P. Libermann, Sur le problème d'équivalence de certaines structures infinitésimales, *Ann. Mat. Pura Appl. (4)* **36** (1954), 27–120. MR 0066020 zbM 0056.15401 2, 28

[29] J. Loftin, Survey on affine spheres, in *Handbook of geometric analysis, No. 2*, *Adv. Lect. Math. (ALM)* **13**, Int. Press, Somerville, MA, 2010, pp. 161–191. MR 2743442 zbM 1214.53013 1

[30] J. C. Loftin, Affine spheres and convex $\mathbb{RP}^n$-manifolds, *Amer. J. Math.* **123** (2001), 255–274. MR 1828223 zbM 0997.53010 1, 2, 4, 28

[31] T. Mettler, Extremal conformal structures on projective surfaces, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **XX** (2020), 1621–1663. DOI 10.2422/2036-2145.201802_008 12, 29

[32] T. Mettler, G. Paternain, Holomorphic differentials, thermostats and Anosov flows, 2019, pp. 553–580. arXiv:1706.03554 5

[33] K. Nomizu, T. Sasaki, *Affine differential geometry*, *Cambridge Tracts in Mathematics* **111**, Cambridge University Press, Cambridge, 1994, Geometry of affine immersions. MR 1311248 zbM 0834.53002 1

[34] M. Struwe, *Variational methods*, fourth ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* **34**, Springer-Verlag, Berlin, 2008. MR 2431434 zbM 1284.49004 27

[35] L. Vrancken, Centroaffine differential geometry and its relations to horizontal submanifolds, in *PDEs, submanifolds and affine differential geometry (Warsaw, 2000)*, *Banach Center Publ.* **57**, Polish Acad. Sci. Inst. Math., Warsaw, 2002, pp. 21–28. DOI 10.4064/bc57-0-3 MR 1972461 zbM 1023.53010 2

[36] C. P. Wang, Some examples of complete hyperbolic affine 2-spheres in $\mathbf{R}^3$, in *Global differential geometry and global analysis (Berlin, 1990)*, *Lecture Notes in Math.* **1481**, Springer, Berlin, 1991, pp. 271–280. DOI 10.1007/BFb0083648 MR 1178538 zbM 0743.53004 2

[37] H. Weyl, Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung., *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* **1921** (1921), 99–112. zbM 48.0844.04 12

Institut für Mathematik, Goethe-Universität Frankfurt, Frankfurt am Main, Germany

Email address: `mettler@math.uni-frankfurt.de`