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Characterizing classical minimal surfaces via the entropy differential

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ABSTRACT. We introduce on any smooth oriented minimal surface in Euclidean 3-space a meromorphic quadratic differential, P, which we call the entropy differential. This differential arises naturally in a number of different contexts. Of particular interest is the realization of its real part as a conservation law for a natural geometric functional – which is, essentially, the entropy of the Gauss curvature. We characterize several classical surfaces – including Enneper's surface, the catenoid and the helicoid – in terms of P. As an application, we prove a novel curvature estimate for embedded minimal surfaces with small entropy differential and an associated compactness theorem.

1. Introduction

Let $\Sigma \subset \mathbb{R}^3$ be a smooth, oriented minimal surface. In this paper, we introduce a meromorphic quadratic differential P on Σ, which we call the *entropy differential*. We use P to characterize several classical surfaces – including Enneper's surface, the catenoid and the helicoid. In particular, subsets of Enneper's surface are the only minimal surfaces on which P vanishes – a fact which we use to prove a novel curvature estimate for embedded minimal surfaces with small entropy differential and an associated compactness result.

The differential P arises naturally in a number of different contexts. Of particular interest is the realization of $T = \operatorname{Re} P$, which we call the *entropy form*, as a conservation law for the diffeomorphism invariant functional

$$\mathcal{E}[g] = \int_\Sigma K_g \log K_g \mu_g .$$

This functional, which is a type of entropy for the curvature, has been previously considered by R. Hamilton in the context of the Ricci flow on surfaces [?]. In particular, we show that if g is a minimal surface metric (i.e. the metric induced by a smooth minimal immersion) for which $K_g \neq 0$, then the metric $\hat{g} = (-K_g)^{3/4} g$ is a critical point of $\mathcal{E}$ with respect to compactly supported conformal deformations. The crucial fact used here is the observation – due to Ricci [?] – that such minimal surface metrics satisfy the so-called *Ricci condition*:

$$\Delta_g \log |K_g| = 4K_g .$$

The differential P also arises as a certain geometric Schwarzian derivative of the Gauss map – a point of view which has antecedents in [?, ?] – and which we will study more thoroughly in a forthcoming paper [?].

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A key observation of the present paper is that, modulo rigid motions, a minimal surface is determined, up to a three-parameter family, by its Hopf differential Q and its entropy differential P. This allows one to characterize several classical minimal surfaces in terms of simple relationships between the Hopf and entropy differentials:

Theorem 4.1. *Let Σ be a smooth oriented non-flat minimal surface in $\mathbb{R}^3$ with entropy differential P. We have:*

(1) *If $P \equiv 0$, then up to a rigid motion and homothety, Σ is contained in Enneper's surface;*
(2) *If $\lambda \neq 0$ and $P \equiv \lambda Q$, then, up to a rigid motion and homothety, Σ is contained in a surface $C \in \mathscr{C}$. If Σ is properly embedded, then it is the catenoid;*
(3) *If $\lambda \neq 0$ and $P \equiv i\lambda Q$, then, up to a rigid motion and homothety, Σ is contained in a surface $H \in \mathscr{H}$. If Σ is properly embedded, then it is the helicoid.*

The families $\mathscr{C}$ and $\mathscr{H}$ are, respectively, the *deformed catenoids* and *deformed helicoids*. These are one parameter families of surfaces containing, respectively, the catenoid and the helicoid – their geometry is discussed thoroughly in **??**.

A consequence of Item (1) of Theorem 4.1 are a family of novel curvature estimates for embedded minimal surfaces. Namely, we introduce a certain family of scale invariant quantities which measure the size of the entropy form and use standard blow-up arguments to derive curvature bounds for embedded surfaces for which these quantities are small. Specifically, for a constant $\alpha > 0$ and smooth minimal surface Σ with entropy form T, we define:

$$||T||_{\Sigma,\alpha} := 2^{\frac{1}{2(1+\alpha)}} \int_\Sigma |T|_g^{\frac{1}{1+\alpha}} |K_g|^{\frac{\alpha}{\alpha+1}} \mu_g.$$

We justify this family by noting that, on the one hand they are scale invariant and, on the other, the "endpoints" are very natural. Indeed,

$$\lim_{\alpha\to\infty} ||T||_{\Sigma,\alpha} = \int_\Sigma |K_g| \mu_g,$$

i.e., one endpoint is the total Gauss curvature, a well studied quantity in minimal surface theory. While,

$$\lim_{\alpha\to 0} ||T||_{\Sigma,\alpha} = \sqrt{2} \int_\Sigma |T|_g \mu_g,$$

i.e., the other endpoint is the L^1 norm of the entropy form which is invariant under the standard action of $\mathrm{PSL}(2,\mathbb{C})$ on the Gauss map of Σ, see [?]. We will not deal directly with this quantity due to the fact that the presence of umbilic points tends to make it infinite.

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