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One-dimensional projective structures, convex curves and the ovals of Benguria and Loss

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ABSTRACT. Benguria and Loss have conjectured that, amongst all smooth closed curves in $\mathbb{R}^2$ of length 2π, the lowest possible eigenvalue of the operator $L = -\Delta + \kappa^2$ is 1. They observed that this value was achieved on a two-parameter family, $\mathcal{O}$, of geometrically distinct ovals containing the round circle and collapsing to a multiplicity-two line segment. We characterize the curves in $\mathcal{O}$ as absolute minima of two related geometric functionals. We also discuss a connection with projective differential geometry and use it to explain the natural symmetries of all three problems.

1. Introduction

In [1], Benguria and Loss conjectured that for any, σ, a smooth closed curve in $\mathbb{R}^2$ of length 2π, the lowest eigenvalue, λ_σ, of the operator $L_\sigma = -\Delta_\sigma + \kappa_\sigma^2$ satisfied $\lambda_\sigma \geq 1$. That is, they conjectured that for all such σ and all functions $f \in H^1(\sigma)$,

$$\int_\sigma |\nabla_\sigma f|^2 + \kappa_\sigma^2 f^2 \, \mathrm{d}s \geq \int_\sigma f^2 \, \mathrm{d}s, \tag{1.1}$$

where $\nabla_\sigma f$ is the intrinsic gradient of f, κ_σ is the geodesic curvature and $\mathrm{d}s$ is the length element. This conjecture was motivated by their observation that it was equivalent to a certain one-dimensional Lieb-Thirring inequality with conjectured sharp constant. They further observed that the above inequality is saturated on a two-parameter family of strictly convex curves which contains the round circle and degenerates into a multiplicity-two line segment. The curves in this family look like ovals and so we call them the ovals of Benguria and Loss and denote the family by $\mathcal{O}$. Finally, they showed that for closed curves $\lambda_\sigma \geq \frac{1}{2}$.

Further work on the conjecture was carried out by Burchard and Thomas in [3]. They showed that λ_σ is strictly minimized in a certain neighborhood of $\mathcal{O}$ in the space of closed curves – verifying the conjecture in this neighborhood. More globally, Linde [5] improved the lower bound to $\lambda_\sigma \geq 0.608$ when σ is a planar convex curves. In addition, he showed that $\lambda_\sigma \geq 1$ when σ satisfied a certain symmetry condition. Recently, Denzler [4] has shown that if the conjecture is false, then the infimum of λ_σ over the space of closed curves is

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achieved by a closed strictly convex planar curve. Coupled with Linde's work, this implies that for any closed curve $\lambda_\sigma \geq 0.608$. In a different direction, the first author and Breiner in [2] connected the conjecture to a certain convexity property for the length of curves in a minimal annulus.

In the present article, we consider the family $\mathcal{O}$ and observe that the curves in this class are the unique minimizers of two natural geometric functionals. To motivate these functionals, we first introduce an energy functional modeled on (1.1). Specifically, for a smooth curve, σ, of length $L(\sigma)$ and function, $f \in C^\infty(\sigma)$, set

$$\mathcal{E}_S[\sigma, f] = \int_\sigma |\nabla_\sigma f|^2 + \kappa_\sigma^2 f^2 - \frac{(2\pi)^2}{L(\sigma)^2} f^2 \, ds. \tag{1.2}$$

Clearly, the conjecture of Benguria and Loss is equivalent to the non-negativity of this functional. For any strictly convex smooth curve, σ, set

$$\mathcal{E}_G[\sigma] = \int_\sigma \frac{|\nabla_\sigma \kappa_\sigma|^2}{4\kappa_\sigma^3} - \frac{(2\pi)^2}{L(\sigma)^2}\frac{1}{\kappa_\sigma} \, ds + 2\pi, \tag{1.3}$$

and

$$\mathcal{E}_G^*[\sigma] = \int_\sigma \frac{|\nabla_\sigma \kappa_\sigma|^2}{4\kappa_\sigma^2} - \kappa_\sigma^2 \, ds + \frac{(2\pi)^2}{L(\sigma)}. \tag{1.4}$$

Notice $\mathcal{E}_G$ is scale invariant, while $\mathcal{E}_G^*$ scales inversely with length. We will show that $\mathcal{E}_G$ and $\mathcal{E}_G^*$ are dual to each other in a certain sense – justifying the notation.

Our main result is that the functionals (1.3) and (1.4) are always non-negative and are zero only for ovals.

Theorem 1.1. *If σ is a smooth strictly convex closed curve in $\mathbb{R}^2$, then both $\mathcal{E}_G[\sigma] \geq 0$ and $\mathcal{E}_G^*[\sigma] \geq 0$ with equality if and only if $\sigma \in \mathcal{O}$.*

To the best of our knowledge both inequalities are new. Clearly,

$$\mathcal{E}_G[\sigma] = \mathcal{E}_S[\sigma, \kappa_\sigma^{-1/2}],$$

and so the non-negativity of (1.3) would follow from the non-negativity of (1.2). Hence, Theorem 1.1 provides evidence for the conjecture of Benguria and Loss.

We also discuss the natural symmetry of these functionals. To do so we need appropriate domains for the functionals. To that end, we say a (possibly open) smooth planar curve is *degree-one* if its unit tangent map is a degree one map from $\mathbb{S}^1$ to $\mathbb{S}^1$ – for instance, any closed convex curve. A degree-one curve is *strictly convex* if the unit tangent map is a diffeomorphism. We show (see §3.3) that there are natural (left and right) actions of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{D}^\infty$, the space of smooth, degree-one curves and on $\mathcal{D}_+^\infty$, the space of smooth strictly convex degree-one curves, which preserve the functionals.

Theorem 1.2. *There are actions of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{D}^\infty \times C^\infty$, the domain of $\mathcal{E}_S$, and on $\mathcal{D}_+^\infty$ the domain of $\mathcal{E}_G$ and $\mathcal{E}_G^*$ so that for $L \in \mathrm{SL}(2,\mathbb{R})$*

$$\mathcal{E}_S[(\sigma, f)\cdot L] = \mathcal{E}_S[\sigma, f], \quad \mathcal{E}_G[\sigma \cdot L] = \mathcal{E}_G[\sigma], \quad \textit{and} \quad \mathcal{E}_G^*[L\cdot\sigma] = \mathcal{E}_G^*[\sigma].$$

Furthermore, there is an involution $*: \mathscr{D}_+^\infty \to \mathscr{D}_+^\infty$ *so that*

$$*(L \cdot \sigma) = *\sigma \cdot L^{-1} \quad \textit{and} \quad \mathscr{E}_G[*\sigma] = \tfrac{L(\sigma)}{2\pi} \mathscr{E}_G^*[\sigma].$$

We observe that $\mathscr{O}$ is precisely the orbit of the round circle under these actions. Generically, the action does not preserve the condition of being a closed curve. Indeed, the image of the set of closed curves under this action is an open set in the space of curves and so is not well suited for the direct method in the calculus of variations. Arguably, this is the source of the difficulty in answering Benguria and Loss's conjecture. Indeed, we prove Theorem 1.1 in part by overcoming it.

2. Preliminaries

Denote by $\mathbb{S}^1 = \{x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$ the unit circle in $\mathbb{R}^2$. Unless otherwise stated, we always assume that $\mathbb{S}^1$ inherits the standard orientation from $\mathbb{R}^2$ and consider $\mathrm{d}\theta$ to be the associated volume form and ∂_θ the dual vector field. Abusing notation slightly, let $\theta : \mathbb{S}^1 \to [0, 2\pi)$ be the compatible chart with $\theta(\mathbf{e}_1) = 0$. Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the covering map so that $\pi^*\mathrm{d}\theta = \mathrm{d}x$ and $\pi(0) = \mathbf{e}_1$ – here x is the usual coordinate on $\mathbb{R}$. Denote by $I : \mathbb{S}^1 \to \mathbb{S}^1$ the involution given by $I(p) = -p$. Hence, $\theta(I(p)) = \theta(p) + \pi \mod 2\pi$.

Definition 2.1. An immersion $\sigma : [0, 2\pi] \to \mathbb{R}^2$ is a *degree-one curve of class* $C^{k+1,\alpha}$, if there is

- a degree-one map $\mathbf{T}_\sigma : \mathbb{S}^1 \to \mathbb{S}^1$ of class $C^{k,\alpha}$, the *unit tangent map of* σ,
- a point $\mathbf{x}_\sigma \in \mathbb{R}^2$, the *base point of* σ, and
- a value $L(\sigma) > 0$, the *length of* σ,

so that

$$\sigma(t) = \mathbf{x}_\sigma + \frac{L(\sigma)}{2\pi} \int_0^t \mathbf{T}_\sigma(\pi(x))\, \mathrm{d}x.$$

The curve σ is *strictly convex* provided the unit tangent map $\mathbf{T}_\sigma$ has a $C^{k,\alpha}$ inverse and is *closed* provided $\sigma(0) = \sigma(2\pi)$.

A degree-one curve, σ, is uniquely determined by the data $(\mathbf{T}_\sigma, \mathbf{x}_\sigma, L(\sigma))$. Denote by $\mathscr{D}^{k+1,\alpha}$ the set of degree-one curves of class $C^{k+1,\alpha}$ and by $\mathscr{D}_+^{k+1,\alpha} \subset \mathscr{D}^{k+1,\alpha}$ the set of strictly convex degree-one curves of of class $C^{k+1,\alpha}$. The length element associated to σ is $\mathrm{d}s = \frac{L(\sigma)}{2\pi}\mathrm{d}x = \frac{L(\sigma)}{2\pi}\pi^*\mathrm{d}\theta = \pi^*\widetilde{\mathrm{d}s}$. If $\sigma \in \mathscr{D}^2$, then the geodesic curvature, κ_σ, of σ satisfies $\kappa_\sigma = \pi^*\tilde{\kappa}_\sigma$ where $\tilde{\kappa}_\sigma \in C^{k-1,\alpha}(\mathbb{S}^1)$ satisfies

$$\int_{\mathbb{S}^1} \tilde{\kappa}_\sigma \, \widetilde{\mathrm{d}s} = 2\pi.$$

Conversely, given such a κ_σ there is a degree-one curve with geodesic curvature κ_σ. Abusing notation slightly, we will not distinguish between $\mathrm{d}s$ and $\widetilde{\mathrm{d}s}$ and between κ_σ and $\tilde{\kappa}_\sigma$. Clearly, $\sigma \in \mathscr{D}_+^2$ if and only if $\kappa_\sigma > 0$.

The *standard parameterization of* $\mathbb{S}^1$ is given by the data $(\mathbf{T}_0, \mathbf{e}_1, 2\pi)$ where

$$\mathbf{T}_0(p) = -\sin(\theta(p))\mathbf{e}_1 + \cos(\theta(p))\mathbf{e}_2.$$

Let $\mathrm{Diff}^{k,\alpha}_+(\mathbb{S}^1)$ denote the orientation preserving diffeomorphisms of $\mathbb{S}^1$ of class $C^{k,\alpha}$ – that is bijective maps of class $C^{k,\alpha}$ with inverse of class $C^{k,\alpha}$. Endow this space with the usual $C^{k,\alpha}$ topology. For $\sigma \in \mathscr{D}^{k+1,\alpha}_+$, we call the map

$$\phi_\sigma = \mathbf{T}_0^{-1} \circ \mathbf{T}_\sigma$$

the *induced diffeomorphism of* σ. Clearly, the induced diffeomorphism of the standard parameterization of $\mathbb{S}^1$ is the identity map.

For $f \in C^k(\mathbb{S}^1)$, let $f' = \partial_\theta f$, $f'' = (f')'$ and likewise for higher order derivatives. Observe that, for $\phi \in \mathrm{Diff}^1_+(\mathbb{S}^1)$, we have $\phi' > 0$ where $\phi' \in C^0(\mathbb{S}^1)$ satisfies $\phi^* \mathrm{d}\theta = \phi' \mathrm{d}\theta$. A simple computation shows that if $\sigma \in \mathscr{D}^2_+$, then $\phi'_\sigma = \kappa_\sigma \frac{L(\sigma)}{2\pi}$.

3. Symmetries of the functionals

Consider the group homomorphism $\Gamma : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ given by

$$\Gamma(L) = \mathbf{x} \mapsto \frac{L \cdot \mathbf{x}}{|L \cdot \mathbf{x}|}$$

where $\mathbf{x} \in \mathbb{S}^1$ and $L \in \mathrm{SL}(2,\mathbb{R})$. Denote the image of Γ by $\mathrm{Möb}(\mathbb{S}^1)$ which we refer to as the Möbius group of $\mathbb{S}^1$. One computes that

$$\mathbf{T}_0 \circ \Gamma(L) = \frac{L \cdot \mathbf{T}_0}{|L \cdot \mathbf{T}_0|}.$$

These are precisely the unit tangent maps of the ovals of [1]. That is,

$$\mathscr{O} = \left\{ \sigma \in \mathscr{D}^\infty_+ : \phi_\sigma \in \mathrm{Möb}(\mathbb{S}^1) \right\}.$$

3.1. The Schwarzian derivative

For $\phi \in \mathrm{Diff}^3_+(\mathbb{S}^1)$ the *Schwarzian derivative* of ϕ is defined as

$$S_\theta(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2,$$

where primes denote derivatives with respect to θ. A fundamental feature of the Schwarzian derivative is that it satisfies the following co-cycle property

$$S_\theta(\phi \circ \psi) = (S_\theta(\phi) \circ \psi) \cdot (\psi')^2 + S_\theta(\psi), \tag{3.1}$$

where $\phi, \psi \in \mathrm{Diff}^3_+(\mathbb{S}^1)$. After some computation, one verifies that the Schwarzian derivative gives the following intrinsic characterization of $\mathrm{Möb}(\mathbb{S}^1)$

$$\phi \in \mathrm{Möb}(\mathbb{S}^1) \iff S_\theta(\phi) + 2(\phi')^2 - 2 = 0. \tag{3.2}$$

The Schwarzian derivative arises most naturally in the context of projective differential geometry. This perspective also gives a conceptual proof of (3.2). For this proof as well as the necessary background the reader may consult the Section B as well as the references cited there.

3.2. Projective Symmetries

We now describe the natural symmetries of (1.2), (1.3) and (1.4). We also introduce a notion of duality for strictly convex degree-one curves – this duality will streamline some of the arguments. For $\sigma \in \mathscr{D}^{k+1,\alpha}$, define the *dual curve*, $\sigma^* \in \mathscr{D}^{k+1,\alpha}$ to be the unique curve with

$$\phi_{\sigma^*} = \phi_\sigma^{-1}, \quad \mathbf{x}_{\sigma^*} = \mathbf{x}_\sigma \quad \text{and} \quad L(\sigma^*) = L(\sigma).$$

That is, σ^* is the curve whose induced diffeomorphism is the inverse to the induced diffeomorphism of σ. Clearly, $(\sigma^*)^* = \sigma$. To proceed further, we note that the functionals (1.3) and (1.4) can, by integrating by parts, be made to naturally involve the Schwarzian derivative. To see this fix $\sigma \in \mathscr{D}^4_+$ with $L(\sigma) = 2\pi$. As $\kappa_\sigma = \phi'_\sigma$,

$$\begin{aligned}\mathcal{E}_G[\sigma] &= \int_{\mathbb{S}^1} \frac{(\phi''_\sigma)^2}{4(\phi'_\sigma)^3} - \frac{1}{\phi'_\sigma} + \phi'_\sigma \, d\theta \\ &= \int_{\mathbb{S}^1} \left(\frac{\phi''_\sigma}{2(\phi'_\sigma)^3}\right)' \phi'_\sigma + \frac{3(\phi''_\sigma)^2}{4(\phi'_\sigma)^3} - \frac{1}{\phi'_\sigma} + \phi'_\sigma \, d\theta \\ &= \frac{1}{2}\int_{\mathbb{S}^1} \frac{S_\theta(\phi_\sigma) + 2(\phi'_\sigma)^2 - 2}{\phi'_\sigma} \, d\theta,\end{aligned} \tag{3.3}$$

where the second equality follows by integrating by parts. Likewise,

$$\mathcal{E}^*_G[\sigma] = -\frac{1}{2}\int_{\mathbb{S}^1} S_\theta(\phi_\sigma) + 2(\phi'_\sigma)^2 - 2 \, d\theta. \tag{3.4}$$

An immediate consequence of this is the following useful fact,

Proposition 3.1. *For $\sigma \in \mathscr{D}^4_+$, we have $\mathcal{E}_G[\sigma] = \frac{L(\sigma)}{2\pi}\mathcal{E}^*_G[\sigma^*]$.*

Proof. By scaling we may assume that $L(\sigma) = 2\pi$. Write $\psi_\sigma = \phi_\sigma^{-1}$. The co-cycle property for the Schwarzian derivative implies that

$$S_\theta(\psi_\sigma) = -\frac{S_\theta(\phi_\sigma) \circ \phi_\sigma^{-1}}{(\phi'_\sigma \circ \phi_\sigma^{-1})^2},$$

where we have used that

$$\phi'_\sigma = \frac{1}{\psi'_\sigma \circ \phi_\sigma}.$$

Hence, by (3.3) and (3.4)

$$\begin{aligned}\mathcal{E}_G[\sigma] &= \frac{1}{2}\int_{\mathbb{S}^1} \frac{S_\theta(\phi_\sigma) + 2(\phi'_\sigma)^2 - 2}{\phi'_\sigma} \, d\theta \\ &= -\frac{1}{2}\int_{\mathbb{S}^1} (S_\theta(\psi_\sigma) \circ \phi_\sigma)\phi'_\sigma + 2(\psi'_\sigma \circ \phi_\sigma)^2 \phi'_\sigma - 2\phi'_\sigma \, d\theta \\ &= -\frac{1}{2}\int_{\mathbb{S}^1} S_\theta(\psi_\sigma) + 2(\psi'_\sigma)^2 - 2 \, d\theta \\ &= \mathcal{E}^*_G[\sigma^*].\end{aligned}$$

□

We now may define the desired actions. Consider first the right action of $\text{Möb}(\mathbb{S}^1)$ on $\mathscr{D}^{k+1,\alpha}$, $\sigma \cdot \varphi = \sigma'$ where $\sigma' \in \mathscr{D}^{k+1,\alpha}$ is the unique element with

$$\mathbf{T}_{\sigma'} = \mathbf{T}_\sigma \circ \varphi, \quad \mathbf{x}_{\sigma'} = \mathbf{x}_\sigma \quad \text{and} \quad L(\sigma') = L(\sigma).$$

Notice, that if $\sigma \in \mathscr{D}_+^{k+1,\alpha}$ is strictly convex, then so is σ' and in this case we have that $\phi_{\sigma'} = \phi_\sigma \circ \varphi$. With this in mind, we also consider a left action of $\mathrm{M\ddot{o}b}(\mathbb{S}^1)$ on $\mathscr{D}_+^{k+1,\alpha}$, $\varphi \cdot \sigma = \sigma'$, where $\sigma' \in \mathscr{D}^{k+1,\alpha}$ is the unique element with

$$\phi_{\sigma'} = \varphi \circ \phi_\sigma, \quad \mathbf{x}_{\sigma'} = \mathbf{x}_\sigma \quad \text{and} \quad L(\sigma') = L(\sigma).$$

We observe that for $\sigma \in \mathscr{D}_+^{k+1}$ and $\varphi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$,

$$\varphi \cdot \sigma^* = (\sigma \cdot \varphi^{-1})^*.$$

Finally, we define a right action of $\mathrm{M\ddot{o}b}(\mathbb{S}^1)$ on $C^{k,\alpha}(\mathbb{S}^1)$ by

$$f \cdot \varphi = (\varphi')^{-1/2} f \circ \varphi.$$

If we use $\mathrm{d}\theta$ to identify $C^\infty(\mathbb{S}^1)$ with $\Omega^{-1/2}(\mathbb{S}^1)$, then this is the natural pull-back action on $\Omega^{-1/2}(\mathbb{S}^1)$ – the space of weight $-1/2$ densities (see Section B).

Proposition 3.2. *For any $\varphi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$, $\sigma \in \mathscr{D}^\infty$ and $f \in C^\infty(\mathbb{S}^1)$,*

$$\mathscr{E}_S[\sigma, f] = \mathscr{E}_S[\sigma \cdot \varphi, f \cdot \varphi].$$

Likewise, for any $\varphi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$ and $\sigma \in \mathscr{D}_+^\infty$,

$$\mathscr{E}_G[\sigma] = \mathscr{E}_G[\sigma \cdot \varphi] \quad \textit{and} \quad \mathscr{E}_G^*[\sigma] = \mathscr{E}_G^*[\varphi \cdot \sigma].$$

Proof. By scaling, it suffices to take $L(\sigma) = 2\pi$ so $\mathrm{d}s = \mathrm{d}\theta$. Set

$$f_\varphi' = (\varphi')^{1/2} f' \circ \varphi - \frac{1}{2} \frac{\varphi''}{(\varphi')^{3/2}} f \circ \varphi.$$

We show the first symmetry by computing,

$$\begin{aligned}
(f_\varphi')^2 &= (f' \circ \varphi)^2 \varphi' - \frac{\varphi''}{\varphi'} (f' \circ \varphi)(f \circ \varphi) + \frac{1}{4} \frac{(\varphi'')^2}{(\varphi')^3} (f \circ \varphi)^2 \\
&= (f' \circ \varphi)^2 \varphi' - \frac{1}{2} \frac{\varphi''}{(\varphi')^2} \partial_\theta (f \circ \varphi)^2 + \frac{1}{4} \frac{(\varphi'')^2}{(\varphi')^2} f_\varphi^2 \\
&= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi''(f \circ \varphi)^2}{2(\varphi')^2} \right) + \left(\frac{\varphi''}{2(\varphi')^2} \right)' (f \circ \varphi)^2 + \frac{1}{4} \frac{(\varphi'')^2}{(\varphi')^2} f_\varphi^2 \\
&= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi''(f \circ \varphi)^2}{2(\varphi')^2} \right) + \frac{1}{2} S_\theta(\varphi) f_\varphi^2 \\
&= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi''(f \circ \varphi)^2}{2(\varphi')^2} \right) + \left(1 - (\varphi')^2\right) f_\varphi^2 \\
&\quad + \frac{1}{2} (S_\theta(\varphi) + 2(\varphi')^2 - 2) f_\varphi^2 \\
&= (f' \circ \varphi)^2 \varphi' - \partial_\theta \left(\frac{\varphi''(f \circ \varphi)^2}{2(\varphi')^2} \right) + \left(1 - (\varphi')^2\right) f_\varphi^2.
\end{aligned}$$

The last equality used $\varphi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$ and (3.2). Integrating by parts gives,

$$\int_{\mathbb{S}^1} (f_\varphi')^2 - f_\varphi^2 \, \mathrm{d}\theta = \int_{\mathbb{S}^1} (f' \circ \varphi)^2 \varphi' - (f \circ \varphi)^2 \varphi' \, \mathrm{d}\theta.$$

Hence, after a change of variables

$$\int_{\mathbb{S}^1} (f'_\varphi)^2 - f_\varphi^2 \, d\theta = \int_{\mathbb{S}^1} (f')^2 - f^2 \, d\theta.$$

Finally,

$$(\kappa_\varphi f_\varphi)^2 = \kappa_\varphi(\varphi(\theta))^2 f \circ \varphi^2 \varphi'$$

and so a change of variables gives,

$$\int_{\mathbb{S}^1} (\kappa_\varphi f_\varphi)^2 \, d\theta = \int_{\mathbb{S}^1} \kappa^2 f^2 \, d\theta.$$

That is, $\mathcal{E}_S[\sigma, f] = \mathcal{E}_S[\sigma \cdot \varphi, f \cdot \varphi]$.

The co-cycle property of the Schwarzian and (3.4) immediately implies

$$\begin{aligned}
\mathcal{E}_G^*[\varphi \cdot \sigma] &= -\frac{1}{2} \int_{\mathbb{S}^1} S_\theta(\varphi \circ \phi_\sigma) + 2((\varphi \circ \phi_\sigma)')^2 - 2 \, d\theta \\
&= -\frac{1}{2} \int_{\mathbb{S}^1} S_\theta(\phi_\sigma) - 2(\phi'_\sigma)^2(\varphi' \circ \phi_\sigma)^2 + 2(\phi'_\sigma)^2 + 2(\phi'_\sigma)^2(\varphi' \circ \phi_\sigma)^2 \\
&\quad - 2 \, d\theta \\
&= \mathcal{E}_G^*[\sigma]
\end{aligned}$$

Finally, using Theorem 3.1

$$\mathcal{E}_G[\sigma \cdot \varphi] = \mathcal{E}_G^*[(\sigma \cdot \varphi)^*] = \mathcal{E}_G^*[\varphi^{-1} \cdot \sigma^*] = \mathcal{E}_G^*[\sigma^*] = \mathcal{E}_G[\sigma].$$

□

Theorem 1.2 is an immediate consequence of Theorem 3.1 and Theorem 3.2 and the fact that $\text{Möb}(\mathbb{S}^1)$ is isomorphic to $\text{SL}(2, \mathbb{R})$.

As a final remark, we observe that we may extend the duality operator to $\mathcal{D}_+^\infty \times C^\infty(\mathbb{S}^1)$ and define a natural dual functional to $\mathcal{E}_S$. Namely, set

$$(\sigma, f)^* = (\sigma^*, f \circ \phi_\sigma^{-1}) \quad \text{and} \quad \mathcal{E}_S^*[\sigma, f] = \int_\sigma \frac{|\nabla_\sigma f|^2}{\kappa_\sigma} - \kappa_\sigma f^2 + \frac{(2\pi)^2}{L(\sigma)^2} \frac{f^2}{\kappa_\sigma} \, ds.$$

We then have,

Proposition 3.3. *If $\sigma \in \mathcal{D}_+^\infty$ and $f \in C^\infty(\mathbb{S}^1)$, then*

$$\mathcal{E}_S[(\sigma, f)^*] = \frac{L(\sigma)}{2\pi} \mathcal{E}_S^*[\sigma, f].$$

Proof. By scaling, we may assume that $L(\sigma) = 2\pi$. Writing $\psi_\sigma = \phi_\sigma^{-1}$, we compute

$$\begin{aligned}
\mathcal{E}_S[(\sigma, f)^*] &= \int_{\mathbb{S}^1} ((f \circ \psi_\sigma)')^2 + (\psi'_\sigma)^2 (f \circ \psi_\sigma)^2 - (f \circ \psi_\sigma)^2 \, d\theta \\
&= \int_{\mathbb{S}^1} (\psi'_\sigma)^2 (f' \circ \psi_\sigma)^2 + (\psi'_\sigma)^2 (f \circ \psi_\sigma)^2 - \frac{(f \circ \psi_\sigma)^2}{\psi'_\sigma} \psi'_\sigma \, d\theta \\
&= \int_{\mathbb{S}^1} (\psi'_\sigma \circ \psi_\sigma^{-1})(f')^2 + (\psi'_\sigma \circ \psi_\sigma^{-1}) f^2 - \frac{f^2}{\psi'_\sigma \circ \psi_\sigma^{-1}} \, d\theta \\
&= \int_{\mathbb{S}^1} \frac{(f')^2}{\phi'_\sigma} + \frac{f^2}{\phi'_\sigma} - \phi'_\sigma f^2 \, d\theta \\
&= \mathcal{E}_S^*[\sigma, f].
\end{aligned}$$

□

4. Deriving the geometric estimates

To prove Theorem 1.1 we will use the direct method in the calculus of variations on an appropriate subclass of the class of degree-one convex curves. This subclass is larger than the class of closed curves. We first note that the conjecture of Benguria and Loss holds for symmetric curves.

Proposition 4.1. *For $\sigma \in \mathcal{D}^2$, if the induced diffeomorphism satisfies $\phi_\sigma \circ I = I \circ \phi_\sigma$, then $\mathcal{E}_S[\sigma, f] \geq 0$ with equality if and only if $\sigma \in \mathcal{O}$ and $f = \kappa_\sigma^{-1/2}$ is the lowest eigenfunction of L_σ.*

Proof. By scaling we may assume $L(\sigma) = 2\pi$. The symmetry implies that $\kappa_\sigma \circ I = \kappa_\sigma$ and $\mathbf{T}_\sigma \circ I = -\mathbf{T}_\sigma$. Hence, $\mathcal{E}_S[\sigma, f] = \mathcal{E}_S[\sigma, f \circ I]$ and so, the variational characterization of the lowest eigenvalue implies that the lowest eigenfunction f must satisfy $f \circ I = f$. As observed in [1],

$$\mathcal{E}_S[\sigma, f] = \int_{\mathbb{S}^1} |\mathbf{y}'|^2 - |\mathbf{y}|^2 \, \mathrm{d}\theta$$

where $\mathbf{y} = f\mathbf{T}_\sigma$. Moreover, $\mathbf{y}(p) = (a\cos\theta(p) + b\sin\theta(p), c\cos\theta(p) + d\sin\theta(p))$ if and only if $\sigma \in \mathcal{O}$. As $\mathbf{y} \circ I = -\mathbf{y}$,

$$\int_{\mathbb{S}^1} \mathbf{y} \, \mathrm{d}\theta = 0$$

and the proposition follows from the one-dimensional Poincaré inequality. □

Corollary 4.2. *For $\sigma \in \mathcal{D}^3_+$, if the induced diffeomorphism satisfies $\phi_\sigma \circ I = I \circ \phi_\sigma$, then $\mathcal{E}_G[\sigma] \geq 0$ with equality if and only if $\sigma \in \mathcal{O}$.*

Proof. Take $f = \kappa_\sigma^{-1/2}$ in (1.2) and use Theorem 4.1. □

Motivated by [5], we make the following definition which is a weakening of the preceding symmetry condition.

Definition 4.3. A point $p \in \mathbb{S}^1$ is a *balance point* of $\phi \in \mathrm{Diff}^0_+$ if $\phi(I(p)) = I(\phi(p))$. Denote the number (possibly infinite) of balance points of ϕ by $n_B(\phi) \in \mathbb{N} \cup \{\infty\}$.

Clearly, if p is a balance point then so is $I(p)$ and so $n_B(\phi)$ is even or ∞. Further, it follows from the intermediate value theorem that $n_B(\phi) \geq 2$.

Our definition of balance point is a slight generalization of Linde's [5] notion of *critical point* for convex closed curves. Indeed, a critical point of a closed convex curve is just a balance point of its induced diffeomorphism. The key observation of Linde [**5, Lemma 2.1**] is that closed convex curves have at least six critical points. We will only need to know that there are at least four critical points and, so, for the sake of completeness, include an adaptation of Linde's argument to show this.

Lemma 4.4. *If $\psi \in \mathrm{Diff}^1_+(\mathbb{S}^1)$ satisfies*

$$\int_{\mathbb{S}^1} \psi' \cos\theta \, \mathrm{d}\theta = \int_{\mathbb{S}^1} \psi' \sin\theta \, \mathrm{d}\theta = 0,$$

then $n_B(\psi) \geq 4$. Hence, if $\sigma \in \mathcal{D}^2_+$ is closed, then $n_B(\phi_\sigma) \geq 4$.

Proof. As $\int_{\mathbb{S}^1} \psi' \, d\theta = 2\pi$ and ψ' is continuous, there is a point p_0, so that if $\gamma_\pm$ are the components of $\mathbb{S}^1 \backslash \{p_0, I(p_0)\}$, then $\int_{\gamma_\pm} \psi' \, d\theta = \pi$. That is, p_0 and $I(p_0)$ are balance points. Expanding ψ' as a Fourier series, rotating so $\theta(p_0) = 0$ and abusing notation slightly, gives that

$$\psi = C + \theta + \sum_{n=2}^{\infty} (a_n \sin n\theta + b_n \cos n\theta) = C + \theta + f(\theta) + g(\theta)$$

where C is a constant, f are the remaining odd terms in the expansion and g are the remaining even terms. By construction, $\psi(0) + \pi = \psi(\pi)$, $f(\theta+\pi) = -f(\theta)$ and $g(\theta+\pi) = g(\theta)$ and so $f(0) = 0 = f(\pi)$ and all balance points of ψ in γ_+ correspond to zeros of f in $(0, \pi)$. If f does not change sign on $(0, \pi)$, then either $f \equiv 0$ and ψ has an infinite number of balance points, or $\int_0^\pi f(\theta) \sin\theta \, d\theta \neq 0$. However, as $f(\theta + \pi)\sin(\theta + \pi) = f(\theta)\sin\theta$, this would imply $\int_0^{2\pi} f(\theta)\sin\theta \, d\theta \neq 0$ which is impossible. Hence, f must change sign and so f has at least one zero in $(0, \pi)$ which verifies the first claim.

To verify the second claim. We first scale so $L(\sigma) = 2\pi$. If σ is closed, then $\int_{\mathbb{S}^1} \mathbf{T}_\sigma \, d\theta = 0$. That is, $\int_{\mathbb{S}^1} \mathbf{T}_0 \circ \phi_\sigma \, d\theta = 0$. Changing variables, gives

$$\int_{\mathbb{S}^1} (\phi_\sigma^{-1})' \mathbf{T}_0 \, d\theta.$$

Hence, $n_B(\phi_\sigma^{-1}) \geq 4$ and it is clear that $n_B(\phi_\sigma) \geq 4$ as well. □

The spaces on which the functionals (1.3) and (1.4) have good lower bounds seem to be spaces of curves whose induced diffeomorphisms have non-trivial number of balance points. Motivated by this, set

$$\mathrm{BDiff}_+^{k,\alpha}(\mathbb{S}^1, N) = \left\{ \phi \in \mathrm{Diff}_+^{k,\alpha}(\mathbb{S}^1) : n_B(\phi) \geq N \right\}.$$

Hence, $\mathrm{BDiff}_+^{k,\alpha}(\mathbb{S}^1, 2) = \mathrm{Diff}_+^{k,\alpha}(\mathbb{S}^1)$ and $\mathrm{Möb}(\mathbb{S}^1) \subset \mathrm{BDiff}_+^{\infty}(\mathbb{S}^1, N)$ for all N. Let $\overline{\mathrm{BDiff}}_+^{k,\alpha}(\mathbb{S}^1, N)$ be the closure of $\mathrm{BDiff}_+^{k,\alpha}(\mathbb{S}^1, N)$ in $\mathrm{Diff}_+^{k,\alpha}(\mathbb{S}^1)$, $\mathrm{B\mathring{D}iff}_+^{k,\alpha}(\mathbb{S}^1, N)$ be the interior and $\partial\mathrm{BDiff}_+^{k,\alpha}(\mathbb{S}^1, N)$ be the topological boundary. The function n_B is not continuous on these spaces. For example, the family $\phi_\lambda \in \mathrm{Diff}_+^\infty(\mathbb{S}^1)$ given by

$$\theta(\phi_\lambda(p)) = 2\cot^{-1}\left(\lambda \cot\left(\frac{1}{2}\theta(p)\right)\right), \tag{4.1}$$

for $\lambda > 0$ has $n_B(\phi_\lambda) = 2$ for $\lambda \neq 1$ and $n_B(\phi_1) = \infty$ and $\phi_\lambda \to \phi_1$ in $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ as $\lambda \to 1$. Likewise, the family $\psi_\tau \in \mathrm{Diff}_+^{1,1}(\mathbb{S}^1)$, for $\tau \in \mathbb{R}$ given by

$$\theta(\psi_\tau(p)) = \begin{cases} \cot^{-1}\left(\tau + \cot(\theta(p) - \frac{\pi}{2})\right) + \frac{\pi}{2} & \theta(p) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ \theta(p) & \theta(p) \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right), \end{cases} \tag{4.2}$$

has $n_B(\psi_\tau) = 2$ for $\tau \neq 0$ and $n_B(\psi_0) = \infty$. Moreover, setting

$$\psi_\tau^\lambda = \phi_\lambda^{-1} \circ \psi_\tau \circ \phi_\lambda \in \mathrm{Diff}_+^{1,1}(\mathbb{S}^1) \tag{4.3}$$

gives a family so that for $\lambda > 1$, $n_B(\psi^\lambda_\tau) = \infty$ and $\psi^\lambda_\tau \to \psi_\tau$ in $\mathrm{Diff}^{1,\alpha}_+(\mathbb{S}^1)$ as $\lambda \to 1$ for any $\alpha \in [0,1)$. Observe that ψ_τ is the extension by the identity of the restriction of an element of $\mathrm{Möb}(\mathbb{S}^1)$ to one component of $\mathbb{S}^1\setminus\{\mathbf{e}_2, -\mathbf{e}_2\}$ and there are no other elements of $\mathrm{Möb}(\mathbb{S}^1)$ for which such an extension exists as an element of $\mathrm{Diff}^1_+(\mathbb{S}^1)$.

The elements of (4.1) show that $\mathrm{Möb}(\mathbb{S}^1) \subset \partial\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$, while the elements of (4.2) show that $\partial\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$ contains elements with $n_B = 2$. In order to proceed further, we must refine the notion of balance point. If $\phi \in \mathrm{Diff}^1_+(\mathbb{S}^1)$, then a balance point p of ϕ is *stable* if and only if $\phi'(p) \neq \phi'(I(p))$ and is *unstable* if $\phi'(p) = \phi'(I(p))$. Denote the number of stable balance points of ϕ by $n_{SB}(\phi)$. For instance, the ψ_τ of (4.2) have $n_{SB}(\psi_\tau) = 0$.

Lemma 4.5. *If $\phi \in \mathrm{Diff}^1_+(\mathbb{S}^1)$, then for each $N \in \mathbb{N}$ there is a C^1 neighborhood, $V = V_N$, of ϕ so that $\min\{n_{SB}(\phi), N\} \leq n_{SB}(\psi)$ for all $\psi \in V$. Furthermore, if ϕ satisfies $n_B(\phi) = n_{SB}(\phi) < \infty$, then n_B is constant in a C^1 neighborhood of ϕ.*

Proof. Let B be the set of balance points of ϕ and $S \subset B$ be the set of stable balance points. It follows from the inverse function theorem that for each $p \in S$, there is an open interval, I_p, in $\mathbb{S}^1$ so that $B \cap I_p = \{p\}$. It is straightforward to show, after fixing smaller open intervals, I'_p, satisfying $p \subset I'_p$ and $\bar{I}'_p \subset I_p$, that there are C^1 neighborhoods, V_p, of ϕ in $\mathrm{Diff}^1_+(\mathbb{S}^1)$ so that all $\psi \in V_p$ have only one stable balance point in I'_p and no unstable balance points.

If $n_{SB}(\phi) > N$, then let $S_N \subset S$ be some choice of N distinct points of S, otherwise, let $S_N = S$. As S_N is finite, $V_N = \cap_{p\in S_N} V_p$ is a an open C^1 neighborhood of ϕ in $\mathrm{Diff}^1_+(\mathbb{S}^1)$ so that for any $\psi \in V_N$, there are $\min\{n_{SB}(\phi), N\}$ stable balance points in $U'_N = \cup_{p\in S_N} I'_p$ and no unstable balance points. Hence, $n_{SB}(\psi) \geq \min\{n_{SB}(\phi), N\}$ which completes the proof of the first claim. The second claim follows by taking $N = n_{SB}(\phi) < \infty$. As $n_B(\phi) = n_{SB}(\phi)$, there are no balance points in $\mathbb{S}^1\setminus U'_N$ and so small C^0 perturbations of ϕ also have no balance points in $\mathbb{S}^1\setminus U'_N$. In other words, by shrinking V_N one can ensure that $n_B(\psi) = n_{SB}(\psi) = n_{SB}(\phi) = n_B(\phi)$ for all $\psi \in V_N$. □

Lemma 4.6. *If $k \geq 1$ and $\phi \in \partial\mathrm{BDiff}^{k,\alpha}_+(\mathbb{S}^1, 4)$, then ϕ has at least one pair of unstable balance points.*

Proof. If $\phi \in \partial\mathrm{BDiff}^{k,\alpha}_+(\mathbb{S}^1, 4)$ for $k \geq 1$, then $\phi \in \partial\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$. Hence, we can restrict attention to the C^1 setting. If $n_B(\phi) = 2$, then the two balance points must be unstable as otherwise Theorem 4.5 would imply that any C^1 perturbation of ϕ also has only two balance points – that is $\phi \notin \overline{\mathrm{BDiff}^1_+}(\mathbb{S}^1, 4)$. If $n_B(\phi) = \infty$, then it must have some unstable balance points, as the set balance points is closed while the set of stable balance points is discrete and so is a proper subset. Finally, if $4 \leq n_B(\phi) < \infty$, then Theorem 4.5 implies at least two of them are unstable. Otherwise, any C^1 perturbation of ϕ would continue to have at least four balance points, i.e., ϕ is in the interior of $\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$. □

We next introduce the appropriate energy space for $\mathcal{E}^*_G$ – we work with this functional as it has nicer analytic properties. It will be convenient to think of $\mathcal{E}^*_G$ as a functional on $\mathrm{Diff}^2_+(\mathbb{S}^1)$ by considering $\mathcal{E}^*_G[\phi] = \mathcal{E}^*_G[\sigma]$ where $\phi = \phi_\sigma$ and $L(\sigma) = 2\pi$. To motivate our choice of energy space set $u = \log \phi' \in C^\infty(\mathbb{S}^1)$. Notice, that u satisfies the non-linear constraint

$$\int_{\mathbb{S}^1} e^u \, \mathrm{d}\theta = 2\pi.$$

A simple change of variables shows that the functional

$$E[u] = \int_{\mathbb{S}^1} \frac{1}{4}(u')^2 - e^{2u} \, \mathrm{d}\theta$$

satisfies $E[u] + 2\pi = \mathcal{E}^*_G[\phi]$. The Euler-Lagrange equation of $E[u]$ with respect to the constraint is a semi-linear ODE of the form

$$\frac{1}{4}u'' + e^{2u} + \beta e^u = 0 \tag{4.4}$$

for some β. Define the following energy space for E

$$H^1_{2\pi}(\mathbb{S}^1) = \left\{ u \in H^1(\mathbb{S}^1) : \int_{\mathbb{S}^1} \mathrm{e}^u \, \mathrm{d}\theta = 2\pi \right\} \subset H^1(\mathbb{S}^1).$$

The Sobolev embedding theorem implies $H^1(\mathbb{S}^1) \subset C^{1/2}(\mathbb{S}^1)$. Hence, $H^1_{2\pi}(\mathbb{S}^1)$ is a closed subset of $H^1(\mathbb{S}^1)$ with respect to the weak topology of $H^1(\mathbb{S}^1)$. Let

$$\mathrm{HDiff}_+(\mathbb{S}^1) = \left\{ \phi \in \mathrm{Diff}^1_+(\mathbb{S}^1) : \log \phi' \in H^1_{2\pi}(\mathbb{S}^1) \right\} \subset \mathrm{Diff}^{1,1/2}_+(\mathbb{S}^1).$$

have a strong (resp. weak) topology determined by $\phi_i \to \phi$ when $\log \phi'_i \to \log \phi'$ strongly in $H^1(\mathbb{S}^1)$ (resp. weakly in $H^1(\mathbb{S}^1)$). Clearly, $\mathcal{E}^*_G$ extends to $\mathrm{HDiff}_+(\mathbb{S}^1)$. As $\mathrm{Diff}^{1,1}_+(\mathbb{S}^1) \subset \mathrm{HDiff}_+(\mathbb{S}^1)$, the family given by (4.2) satisfies $\psi_\tau \in \mathrm{HDiff}_+(\mathbb{S}^1)$ and one computes that $\mathcal{E}^*_G[\psi_\tau] = 0$.

We will need the following smoothing lemma:

Lemma 4.7. *For $\phi \in \mathrm{HDiff}_+(\mathbb{S}^1)$, there exists a sequence $\phi_i \in \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ with $\phi_i \to \phi$ in the strong topology of $\mathrm{HDiff}_+(\mathbb{S}^1)$. Furthermore, if ϕ satisfies $\phi \circ I = I \circ \phi$, then the ϕ_i may be chosen so $\phi_i \circ I = I \circ \phi_i$.*

Proof. Fix $p_0 \in \mathbb{S}^1$, let $\nu_\epsilon(p, p_0)$ be a family of C^∞ mollifiers with $\nu_\epsilon(p, p_0) \geq 0$, $\mathrm{supp}(\nu_\epsilon(\cdot, p_0)) \subset B_\epsilon(p_0)$, $\nu_\epsilon(p, p_0) = \nu_\epsilon(p_0, p)$, $\nu_\epsilon(I(p_0), I(p)) = \nu_\epsilon(p_0, p)$ and $\int_{\mathbb{S}^1} \nu_\epsilon(p, p_0) \, \mathrm{d}\theta(p) = 1$. That is, $\lim_{\epsilon \to 0} \nu_\epsilon(p, p_0) = \delta_{p_0}(p)$ the Dirac delta with mass at p_0. Set

$$P_\epsilon = \int_{\mathbb{S}^1} \nu_\epsilon(\cdot, p)\phi'(p) \, \mathrm{d}\theta(p) \in C^\infty(\mathbb{S}^1).$$

Hence, $\int_{\mathbb{S}^1} P_\epsilon \, \mathrm{d}\theta = 2\pi$ and $P_\epsilon \geq \min_{\mathbb{S}^1} \phi' > 0$. It follows, that there are $\phi_\epsilon \in \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ so that $\phi_\epsilon(p_0) = \phi(p_0)$ and $\phi'_\epsilon = P_\epsilon$. As $\log \phi' \in H^1(\mathbb{S}^1)$, $\phi' \in H^1(\mathbb{S}^1)$ and so $P_\epsilon \to \phi'$ strongly in $H^1(\mathbb{S}^1)$. This convergence together with the uniform lower bound on P_ϵ and the Sobolev embedding theorem implies that $\log P_\epsilon$ converge strongly in $H^1(\mathbb{S}^1)$ to $\log \phi'$ – that is, $\phi_\epsilon \to \phi$ strongly in $\mathrm{HDiff}_+(\mathbb{S}^1)$.

Finally, we observe that if $\phi \circ I = I \circ \phi$, then $\phi' \circ I = \phi'$ and so $P_\epsilon \circ I = P_\epsilon$. In particular, if $\phi \circ I = I \circ \phi$, then $\phi_\epsilon \circ I = I \circ \phi_\epsilon$. ∎

Lemma 4.8. *If $\phi \in \mathrm{HDiff}_+(\mathbb{S}^1)$ satisfies $\phi \circ I = I \circ \phi$, then $\mathcal{E}^*_G[\phi] \geq 0$ with equality if and only if $\phi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$.*

Proof. By Theorem 4.7, there are a sequence of $\phi_i \in \mathrm{Diff}^\infty_+(\mathbb{S}^1)$, with $\phi_i \circ I = I \circ \phi_i$ and $\phi_i \to \phi$ strongly in $\mathrm{HDiff}_+(\mathbb{S}^1)$. In particular, $\mathcal{E}^*_G[\phi_i] \to \mathcal{E}^*_G[\phi]$. Set $\psi_i = \phi_i^{-1}$ and note that $\psi_i \circ I = I \circ \psi_i$. Further, let $\sigma_i \in \mathcal{D}^+_i$ have induced diffeomorphism ψ_i. By (3.4), Theorem 3.1 and Theorem 4.2,

$$\mathcal{E}^*_G[\phi_i] = \mathcal{E}^*_G[\sigma^*_i] = \mathcal{E}_G[\sigma_i] \geq 0,$$

proving the desired inequality. If one has equality, then the inequality implies that ϕ is critical with respect to variations preserving the symmetry. It follows that ϕ is smooth and so $\phi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$ by Theorem 4.2. □

A symmetrization argument and Theorem 4.8 imply:

Proposition 4.9. *If $\phi \in \mathrm{HDiff}_+(\mathbb{S}^1) \cap \partial\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$, then*

$$\mathcal{E}^*_G[\phi] \geq 0$$

with equality if and only if $\phi = \psi_\tau \circ \hat{\phi}$ where ψ_τ is of the form (4.2) for some $\tau \in \mathbb{R}$ and $\hat{\phi} \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$.

Proof. Let $\phi \in \mathrm{HDiff}^+(\mathbb{S}^1) \cap \partial\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$. As $\phi \in \mathrm{Diff}^{1,\frac{1}{2}}_+(\mathbb{S}^1)$, Theorem 4.6 implies that ϕ has at least one unstable balance point p_0. Let $\gamma_\pm$ be the two components of $\mathbb{S}^1 \setminus \{p_0, I(p_0)\}$. Up to relabeling, we may assume that

$$\mathcal{E}^*_G[\phi] \geq 2\left(\int_{\gamma_-} \frac{1}{4}(u')^2 - \mathrm{e}^{2u}\, \mathrm{d}\theta + 2\pi\right)$$

where $u = \log \phi'$. Now define

$$\tilde{u}(p) = \begin{cases} u(p) & p \in \bar{\gamma}_- \\ u(I(p)) & p \in \gamma_+ \end{cases}$$

Here, $\bar{\gamma}_-$ is the closure of γ_- in $\mathbb{S}^1$. Clearly, $\tilde{u}$ is continuous, $\int_{\mathbb{S}^1} \mathrm{e}^{\tilde{u}}\, \mathrm{d}\theta = 2\pi$ and

$$\mathcal{E}^*_G[\phi] \geq E[\tilde{u}] + 2\pi.$$

Hence, there is a $\tilde{\phi} \in \mathrm{Diff}^{1,1}_+(\mathbb{S}^1) \subset \mathrm{HDiff}_+(\mathbb{S}^1)$ so that $\tilde{u} = \log \tilde{\phi}'$. By construction, $\tilde{\phi} \circ I = I \circ \tilde{\phi}$ and so by Theorem 4.8

$$\mathcal{E}^*_G[\phi] \geq \mathcal{E}^*_G[\tilde{\phi}] \geq 0,$$

with equality if and only if $\tilde{\phi} \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$.

In the case of equality for ϕ we could reflect either γ_+ or γ_-, hence the preceding argument implies, $\phi|_{\gamma_\pm} = \phi_\pm$ for $\phi_\pm \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$ which satisfy $\phi_\pm(\gamma_+) = \gamma_+$. By precomposing with a rotation, we may assume that $\{p_0, I(p_0)\} = \{\mathbf{e}_2, -\mathbf{e}_2\}$ and $\theta(\gamma_+) = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Taking $\hat{\phi} = \phi_- \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$, one has $\phi \circ \hat{\phi}^{-1} \in \mathrm{Diff}^1_+(\mathbb{S}^1)$ and is the identity map on γ_- and some element of $\mathrm{M\ddot{o}b}(\mathbb{S}^1)$ on γ_+. This implies that $\phi \circ \hat{\phi}^{-1} = \psi_\tau$ where ψ_τ is of the form (4.2) for some $\tau \in \mathbb{R}$. That is, $\phi = \psi_\tau \circ \hat{\phi}$. □

We next analyze certain ODEs generalizing (4.4).

Proposition 4.10. *Fix $\gamma \geq 2\pi$. If $u \in C^\infty(\mathbb{S}^1)$ satisfies the ODE*

$$\frac{1}{4}u'' - \alpha e^{2u} + \beta e^{u} = 0 \tag{4.5}$$

and the constraints

$$\int_{\mathbb{S}^1} e^u \, d\theta = 2\pi \text{ and } \int_{\mathbb{S}^1} e^{2u} \, d\theta = \gamma, \tag{4.6}$$

then either $\gamma = 2\pi$, $\alpha = \beta$ and $u \equiv 0$ or $\gamma > 2\pi$ and there is an $n \in \mathbb{N}$ so that $\alpha = -n^2$ and $\beta = -\frac{\gamma}{2\pi}n^2$ and

$$u(p) = -\log\left(\frac{\gamma}{2\pi} + \sqrt{\left(\frac{\gamma}{2\pi}\right)^2 - 1}\cos(n(\theta(p) - \theta_0))\right)$$

for some θ_0. In this case,

$$E[u] = -2\pi\frac{n^2}{4} + \frac{(n^2-4)}{4}\gamma.$$

Hence, if $n \geq 2$, then

$$E[u] \geq -2\pi.$$

with equality if and only if $\gamma = 2\pi$ or $n = 2$.

Proof. It is straightforward to see that (4.5) has the conservation law

$$\frac{1}{4}(u')^2 - \alpha e^{2u} + 2\beta e^u = \eta.$$

Integrating this we see that

$$E[u] + (1-\alpha)\gamma + 4\pi\beta = 2\pi\eta.$$

However, integrating (4.5) gives that

$$-\alpha\gamma + 2\pi\beta = 0$$

and hence

$$E[u] = 2\pi\eta - \gamma - 2\pi\beta.$$

Now set $U = e^{-u}$ one has that

$$\frac{1}{4}U'' = -\frac{1}{4}e^{-u}u'' + \frac{1}{4}e^{-u}(u')^2 = -\alpha e^u + \beta + \alpha e^u - 2\beta + \eta e^{-u} = \eta U - \beta$$

That is, U satisfies

$$U'' - 4\eta U = -4\beta.$$

As $U \in C^\infty(\mathbb{S}^1)$, either $U = \frac{\beta}{\eta}$, or $4\eta = -n^2$ for some $n \in \mathbb{Z}^+$ and

$$U = \frac{\beta}{\eta} + C_1 \cos\sqrt{-4\eta}\theta + C_2 \sin\sqrt{-4\eta}\theta$$

for some constants C_1, C_2. In the first case, the constraints force $\eta = \beta$ and so $u = 0$, $\alpha = \beta = \eta$, $\gamma = 2\pi$ and $E = -2\pi$.

In the second case, we first note that $U > 0$ and so

$$\frac{\beta}{\eta} > \sqrt{C_1^2 + C_2^2}.$$

Using the calculus of residues, we compute that

$$\begin{aligned}\int_{\mathbb{S}^1} \mathrm{e}^u \, \mathrm{d}\theta &= \int_{\mathbb{S}^1} \frac{1}{U} \, \mathrm{d}\theta \\ &= \int_{\mathbb{S}^1} \frac{1}{\frac{\beta}{\eta} + \frac{C_1}{2}(z^n + z^{-n}) + \frac{C_2}{2i}(z^n - z^{-n})} \frac{\mathrm{d}z}{iz} \\ &= \frac{2\pi}{\sqrt{\left(\frac{\beta}{\eta}\right)^2 - C_1^2 - C_2^2}}.\end{aligned}$$

Keeping in mind that $U > 0$, the first constraint is satisfied if and only if

$$\beta = \eta\sqrt{1 + C_1^2 + C_2^2}.$$

Hence,

$$u = -\log\left(\sqrt{1 + C_1^2 + C_2^2} + C_1 \cos\sqrt{-4\eta}\theta + C_2 \sin\sqrt{-4\eta}\theta\right).$$

Plugging this into (4.5), shows that $\alpha = \eta$. Hence,

$$\gamma = 2\pi\beta/\alpha = 2\pi\sqrt{1 + C_2^2 + C_2^2}.$$

We conclude that,

$$E[u] = 2\pi\eta - \gamma - \eta\gamma = -2\pi\frac{n^2}{4} + (\frac{1}{4}n^2 - 1)\gamma.$$

Hence, if $n \geq 2$, then as $\gamma \geq 2\pi$

$$E[u] \geq -2\pi\frac{n^2}{4} + 2\pi(\frac{n^2}{4} - 1) \geq -2\pi.$$

with equality if and only if $n = 2$. □

Remark 4.11. If $n = 1$, then as $\gamma \to \infty$, $E[u] \to 0$. If $n = 2$, then $u = \log\phi'$ for $\phi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$.

Combining Theorem 4.9 and Theorem 4.10 gives:

Proposition 4.12. *If* $\phi \in \mathrm{HDiff}_+(\mathbb{S}^1) \cap \overline{\mathrm{BDiff}}^1_+(\mathbb{S}^1, 4)$, *then*

$$\mathcal{E}^*_G[\phi] \geq 0$$

with equality if and only if $\phi = \psi_\tau \circ \hat{\phi}$ *where* ψ_τ *is of the form (4.2) for some* $\tau \in \mathbb{R}$ *and* $\hat{\phi} \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$. *If, in addition,* $\phi \in \mathrm{Diff}^2_+(\mathbb{S}^1)$ *or* $\phi \in \mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$, *then equality occurs if and only if* $\phi \in \mathrm{M\ddot{o}b}(\mathbb{S}^1)$.

Remark 4.13. This result is sharp in that the inequality fails for (4.1).

Proof. If inequality does not hold, then there is a $\phi_0 \in \mathrm{HDiff}_+(\mathbb{S}^1) \cap \overline{\mathrm{BDiff}}^1_+(\mathbb{S}^1, 4)$ so that $\mathcal{E}^*_G[\phi_0] < 0$. Let $u_0 = \log\phi_0'$ and set $\gamma_0 = \int_{\mathbb{S}^1}(\phi')^2 \, \mathrm{d}\theta = \int_{\mathbb{S}^1} \mathrm{e}^{2u_0} \, \mathrm{d}\theta$. The Cauchy-Schwarz inequality implies that $\gamma_0 \geq 2\pi$ with equality if and only if $u_0 \equiv 0$. Now consider the minimization problem

$$E(\gamma) = \inf\left\{\mathcal{E}^*_G[\phi] \; \phi \in \mathrm{HDiff}_+(\mathbb{S}^1) \cap \overline{\mathrm{BDiff}}^1_+(\mathbb{S}^1, 4), \int_{\mathbb{S}^1}(\phi')^2 \, \mathrm{d}\theta = \gamma\right\}. \tag{4.7}$$

Clearly, our assumption ensures that $E(\gamma_0) \leq \mathcal{E}^*_G[\phi_0] < 0$. Notice without the constraint $\int_{\mathbb{S}^1}(\phi')^2 \, \mathrm{d}\theta$ the symmetry of Theorem 1.2 would imply that E

is not coercive for the H^1-norm of $u = \log \phi'$. However, with the constraint we are minimizing the Dirichlet energy of u and so the Rellich compactness theorem gives a $u_{\min} \in H^1_{2\pi}(\mathbb{S}^1)$ satisfying

$$E(\gamma_0) = \int_{\mathbb{S}^1} \frac{1}{4}(u'_{\min})^2 - e^{2u_{\min}}\, d\theta + 2\pi = \int_{\mathbb{S}^1} \frac{1}{4}(u'_{\min})^2\, d\theta - \gamma_0 + 2\pi < 0.$$

and, hence, a $\phi_{\min} \in \mathrm{HDiff}_+(\mathbb{S}^1) \cap \overline{\mathrm{BDiff}}^1_+(\mathbb{S}^1, 4)$ so that $\log \phi'_{\min} = u_{\min}$. However, Theorem 4.9 implies that $\phi_{\min} \in \mathrm{B}\mathring{\mathrm{D}}\mathrm{iff}^1_+(\mathbb{S}^1, 4)$. This implies that $u_{\min}$ is critical with respect to arbitrary variations in $H^1(\mathbb{S}^1)$ which preserve the constraints

$$\textstyle\int_{\mathbb{S}^1} e^u\, d\theta = 2\pi \quad \text{and} \quad \int_{\mathbb{S}^1} e^{2u}\, d\theta = \gamma_0.$$

Hence, $u_{\min}$ weakly satisfies the Euler-Lagrange equation

$$\frac{1}{4}u''_{\min} - \alpha e^{2u_{\min}} + \beta e^{u_{\min}} = 0.$$

As this is a semi-linear ODE and $u_{\min} \in C^{1/2}(\mathbb{S}^1)$ by Sobolev embedding, $u_{\min} \in C^{2+\alpha}(\mathbb{S}^1)$ and satisfies this equation classically. Hence, $u_{\min}$ is smooth by standard ODE theory. Notice, that if ϕ_λ is one of the elements of (4.1), then

$$u_\lambda = \log \phi'_\lambda = -\log\left(\frac{1}{2}(\lambda + \lambda^{-1}) + \frac{1}{2}(\lambda - \lambda^{-1})\cos\theta(p)\right).$$

Applying, Theorem 4.10 to $u_{\min}$ we see that, up to a rotation, if $n = 1$, then $\phi_{\min} = \phi_\lambda$ for some λ. As $n_B(\phi_{\min}) \geq 4$, this is impossible. Hence, $n \geq 2$ and so $E[u_{\min}] \geq 0$ which contradicts $E(\gamma_0) < 0$ and proves the inequality.

Equality cannot hold for $\phi \in \mathrm{B}\mathring{\mathrm{D}}\mathrm{iff}^1_+(\mathbb{S}^1, 4)$. If it did, ϕ would be a critical point for $\mathcal{E}^*_G$ with respect to arbitrary variations in $\mathrm{HDiff}_+(\mathbb{S}^1)$. Applying Theorem 4.10 to $u = \log \phi'$ shows this is impossible. Hence, equality is only achieved on $\partial\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$ and so the claim follows from Theorem 4.9 and the observation that, for ψ_τ as in (4.2), $\psi_\tau \in \mathrm{Diff}^2_+(\mathbb{S}^1)$ or $\mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$ if and only if $\tau = 0$. □

We may now conclude the main geometric estimates.

Proof of Theorem 1.1. The natural scaling of the problem means that we may apply a homothety to take $L(\sigma) = 2\pi$. As σ is a smooth closed strictly convex curve, it is a smooth degree-one strictly convex curve. Let $\phi_\sigma \in \mathrm{Diff}^+(\mathbb{S}^1)$, be the induced diffeomorphism and let $\psi_\sigma = \phi_\sigma^{-1}$. By Theorem 4.4, $\phi_\sigma, \psi_\sigma \in \mathrm{BDiff}^1_+(\mathbb{S}^1, 4)$. The claim now follows from Propositions Theorem 4.12 and Theorem 3.1. □

Appendix A. On extending the conjecture of Benguria and Loss

Benguria and Loss's conjecture concerns closed curves. In light of the present paper, specifically the symmetry of Theorem 1.2, it is tempting to think that their conjecture can be extended to degree-one curves with more than two balance points. However, this is not the case.

Lemma A.1. *For every $N \in \mathbb{N}$, there is a $\sigma \in \mathscr{D}_+^\infty$ so that $\phi_\sigma \in \mathrm{BDiff}_+^\infty(\mathbb{S}^1, N)$ and $\mathcal{E}_S[\sigma, f] < 0$ for some function $f \in C^\infty(\mathbb{S}^1)$.*

Proof. Consider σ_τ to be the curve in $\mathscr{D}_+^{2,1}$ which has $\sigma_\tau(\mathbf{e}_1) = \mathbf{e}_1$, $L(\sigma_\tau) = 2\pi$ and induced diffeomorphism $\phi_{\sigma_\tau} = \psi_\tau$ where ψ_τ is given by (4.2). Note, that for $\tau \neq 0$, σ_τ is not closed. One computes that for $f_\tau = \kappa_{\sigma_\tau}^{-1/2} \in C^{0,1}(\mathbb{S}^1) \subset H^1(\mathbb{S}^1)$, that $\mathcal{E}_S[\sigma_\tau, f_\tau] = \mathcal{E}_G[\sigma_\tau] = 0$. However,

$$L_{\sigma_\tau} f_\tau = -f_\tau'' + \kappa_{\sigma_\tau}^2 f_\tau = f_\tau + C(\tau)\delta_{\mathbf{e}_2} - C(\tau)\delta_{-\mathbf{e}_2},$$

distributionally and the constant $C(\tau) \neq 0$ if and only if $\tau \neq 0$. Hence, for $\tau \neq 0$, f_τ is not an eigenfunction and so there must be a $\hat{f}_\tau \in C^2(\mathbb{S}^1)$ with $\mathcal{E}_S[\sigma_\tau, \hat{f}_\tau] < 0$. Consider the elements $\psi_\tau^\lambda \in \mathrm{Diff}_+^{1,1}(\mathbb{S}^1)$ given by (4.3) and pick $\sigma_\tau^\lambda \in \mathscr{D}_+^{2,1}$ so that $\sigma_\tau^\lambda(\mathbf{e}_1) = \mathbf{e}_1$, $L(\sigma_\tau^\lambda) = 2\pi$ and the induced diffeomorphism is ψ_τ^λ. Clearly, $\sigma_\tau^\lambda \to \sigma_\tau$ as $\lambda \to 1$ in the C^2 topology. Hence, $\mathcal{E}_S[\sigma_\tau^\lambda, \hat{f}_\tau] \to \mathcal{E}_S[\sigma_\tau, \hat{f}_\tau]$ as $\lambda \to 1$. Hence, for $\tau \neq 0$ and $\lambda > 1$ sufficiently close to 1, we obtain a $\sigma \in \mathscr{D}_+^\infty$ with $n_B(\sigma) = \infty$ and $\mathcal{E}_S[\sigma, \hat{f}_\tau] < 0$ by smoothing out σ_τ^λ as in Theorem 4.7. Smoothing out $\hat{f}_\tau$ gives f so that $\mathcal{E}_S[\sigma, f] < 0$. □

Appendix B. Projective structures

We review some basic concepts from projective differential geometry which will motivate the definition of $\mathrm{Möb}(\mathbb{S}^1)$ made above as well as provide the natural context for the symmetries of the functionals of (1.2), (1.3) and (1.4). This is a vast subject with many different perspectives and we present only a summarized version. We refer the interested reader to the excellent book [6] by Ovsienko and Tabachnikov as well as their article [7] – these were our main sources for this material.

B.1. One-Dimensional Projective Differential Geometry

Let M be a one-dimensional oriented manifold. We fix a square root $(T^*M)^{1/2}$ of the cotangent bundle of M so that we have an isomorphism of line bundles

$$(T^*M)^{1/2} \otimes (T^*M)^{1/2} \simeq T^*M.$$

Remark B.1. Note that on the circle there are two non-isomorphic choices of such a root, the trivial line bundle and the Möbius strip. In what follows we will work with the trivial root on the circle.

For an integer ℓ we denote by $\Omega^{\ell/2}(M)$ the space of smooth densities of weight $\ell/2$ on M. That is, an element in $\Omega^{\ell/2}(M)$ is a smooth section of the ℓ-th tensorial power of $(T^*M)^{1/2}$. As usual, for $\ell < 0$ we define

$$\left((T^*M)^{1/2}\right)^{\otimes \ell} = \left((TM)^{1/2}\right)^{\otimes(-\ell)}$$

where $(TM)^{1/2}$ denotes the dual bundle of $(T^*M)^{1/2}$.

Note that an affine connection ∇ on $TM \simeq (T^*M)^{-1}$ induces a connection on all tensorial powers of $(T^*M)^{1/2}$. By standard abuse of notation, we will denote these connections by ∇ as well. In particular, we have first order differential operators

$$\nabla : \Omega^{\ell/2}(M) \to \Omega^{\ell/2+1}(M).$$

A *real projective structure*, $\mathscr{P}$ on M is a second-order elliptic differential operator

$$\mathscr{P} : \Omega^{-1/2}(M) \to \Omega^{3/2}(M)$$

so that there is some affine connection ∇ on M and $P \in \Omega^2(M)$ with

$$\mathscr{P} = \nabla^2 + P.$$

One verifies that, given two real projective structures $\mathscr{P}_1$ and $\mathscr{P}_2$, $\mathscr{P}_2 - \mathscr{P}_1 \in \Omega^2(M)$ is a zero-order operator. Hence, the space of real projective structures is an affine space with associated vector space $\Omega^2(M)$. Given an orientation preserving smooth diffeomorphism $\phi : M_1 \to M_2$ we define the push forward and pull back of real projective structures $\mathscr{P}_i$ on M_i in an obvious fashion. That is,

$$(\phi_* \mathscr{P}_1) \cdot \theta = (\phi^{-1})^* \left(\mathscr{P}_1 \cdot \phi^* \theta\right) \quad \text{and} \quad (\phi^* \mathscr{P}_2) \cdot \theta = \phi^* \left(\mathscr{P}_2 \cdot (\phi^{-1})^* \theta\right).$$

The *Schwarzian derivative* of ϕ relative to $\mathscr{P}_1, \mathscr{P}_2$ is

$$S_{\mathscr{P}_1, \mathscr{P}_2}(\phi) = \phi^* \mathscr{P}_2 - \mathscr{P}_1 \in \Omega^2(M_1).$$

The Schwarzian satisfies the following co-cycle condition

$$S_{\mathscr{P}_1, \mathscr{P}_3}(\phi_2 \circ \phi_1) = \phi_1^* S_{\mathscr{P}_2, \mathscr{P}_3}(\phi_2) + S_{\mathscr{P}_1, \mathscr{P}_2}(\phi_1). \tag{B.1}$$

Given a $\phi \in \mathrm{Diff}^\infty_+(M)$ and a real projective structure $\mathscr{P}$ write $S_{\mathscr{P}}(\phi) = S_{\mathscr{P}, \mathscr{P}}(\phi)$. An orientation preserving diffeomorphism ϕ is a *Möbius transformation of* $\mathscr{P}$ if and only if $S_{\mathscr{P}}(\phi) = 0$. The co-cycle condition implies that the set of such maps forms a subgroup, $\mathrm{Möb}(\mathscr{P})$, of $\mathrm{Diff}^\infty_+(M)$.

Let $\mathbb{RP}^1$ be the one-dimensional real projective space – in other words the space of unoriented lines through the origin in $\mathbb{R}^2$. Let (x_1, x_2) be the usual linear coordinates on $\mathbb{R}^2$. If $(x_1, x_2) \neq 0$, then we denote by $[x_1 : x_2]$ the point in $\mathbb{RP}^1$ corresponding to the line through the origin and (x_1, x_2). On the chart $U = \{[x_1, x_2] : x_2 \neq 0\}$ we have the affine coordinate $\tau = x_1/x_2$ for $\mathbb{RP}^1$. Let $\tau\nabla$ be the (unique) connection so that ∂_τ is parallel. There is a unique real projective structure $\mathscr{P}_{\mathbb{RP}^1}$ on $\mathbb{RP}^1$ so that $\mathscr{P}_{\mathbb{RP}^1} = \tau\nabla^2$. This is the *standard real projective structure* on $\mathbb{RP}^1$.

If $\phi \in \mathrm{Diff}^\infty_+(\mathbb{RP}^1)$, then one computes that

$$S_{\mathbb{RP}^1}(\phi) = S_{\mathscr{P}_{\mathbb{RP}^1}}(\phi) = \left(\frac{\phi'''}{\phi'} - \frac{3}{2} \left(\frac{\phi''}{\phi'} \right)^2 \right) \mathrm{d}\tau^2$$

where here $\phi' = \partial_\tau(\tau \circ \phi)$ and likewise for the higher derivatives. This is the classical form of the Schwarzian derivative introduced in §3. Write $\mathrm{Möb}(\mathbb{RP}^1)$ for the Möbius group of $\mathscr{P}_{\mathbb{RP}^1}$ and observe these are the fractional linear transformations. Indeed, if $\phi \in \mathrm{Möb}(\mathbb{RP}^1)$, then there is a matrix

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

so that

$$\tau(\phi(p)) = \frac{a\tau(p)+b}{c\tau(p)+d}.$$

This corresponds to the natural action of $\mathrm{SL}(2,\mathbb{R})$ on the space of lines through the origin. Let

$$\gamma : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Diff}^{\infty}_{+}(\mathbb{RP}^1).$$

denote this group homomorphism. Notice that $\ker\rho = \pm\mathrm{Id}$ and so this map induces an injective homomorphism

$$\tilde{\gamma} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Diff}^{\infty}_{+}(\mathbb{RP}^1)$$

whose image is $\mathrm{Möb}(\mathbb{RP}^1)$.

Consider the natural map $T : \mathbb{S}^1 \to \mathbb{RP}^1$ given by sending a point p to the tangent line to $\mathbb{S}^1$ through p. Let ${}^{\theta}\nabla$ be the unique connection on $\mathbb{S}^1$ so that ∂_θ is parallel and let $\mathscr{P}_{\mathbb{S}^1} = {}^{\theta}\nabla^2$. If $\phi \in \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$, then one computes that

$$S_{\mathbb{S}^1}(\phi) = S_{\mathscr{P}_{\mathbb{S}^1}}(\phi) = \left(\frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2\right)\mathrm{d}\theta^2$$

where here ϕ' has already been defined. Define $S_\theta(\phi)$ so $S_{\mathbb{S}^1}(\phi) = S_\theta(\phi)\mathrm{d}\theta^2$.

As $T \circ I = T$, if $\phi \in \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ satisfies $\phi \circ I = I \circ \phi$, then there is a well-defined element $\tilde{T}(\phi) \in \mathrm{Diff}^{\infty}_{+}(\mathbb{RP}^1)$ so that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\phi} & \mathbb{S}^1 \\ \big\downarrow T & & \big\downarrow T \\ \mathbb{RP}^1 & \xrightarrow{\tilde{T}(\phi)} & \mathbb{RP}^1 \end{array}$$

A straightforward computation shows that,

$$S_{\mathbb{S}^1,\mathbb{RP}^1}(T) = S_{\mathscr{P}_{\mathbb{S}^1},\mathscr{P}_{\mathbb{RP}^1}}(T) = 2\mathrm{d}\theta^2.$$

Hence, for a $\phi \in \mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ which satisfies $\phi \circ I = I \circ \phi$ the co-cycle relation for the Schwarzian implies

$$\begin{aligned} 0 &= S_{\mathbb{S}^1,\mathbb{RP}^1}(T\circ\phi) - S_{\mathbb{S}^1,\mathbb{RP}^1}(\tilde{T}(\phi)\circ T) \\ &= 2\phi^*\mathrm{d}\theta^2 + S_{\mathbb{S}^1}(\phi) - T^*S_{\mathbb{RP}^1}(\tilde{T}(\phi)) + 2\mathrm{d}\theta^2 \\ &= S_{\mathbb{S}^1}(\phi) + 2(\phi')^2\mathrm{d}\theta^2 + 2\mathrm{d}\theta^2 - T^*S_{\mathbb{RP}^1}(\tilde{T}(\phi)) \end{aligned}$$

That is,

$$S_{\mathbb{S}^1}(\phi) + 2(\phi')^2\mathrm{d}\theta^2 - 2\mathrm{d}\theta^2 = T^*S_{\mathbb{RP}^1}(\tilde{T}(\phi)).$$

One verifies from their definitions that $\tilde{T}(\mathrm{Möb}(\mathbb{S}^1)) = \mathrm{Möb}(\mathbb{RP}^1)$ and which gives (3.2). Finally, we note the following commutative diagram

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\Gamma} & \mathrm{Mob}(\mathbb{S}^1) \\ \big\downarrow \pi & & \big\downarrow \tilde{T} \\ \mathrm{PSL}(2,\mathbb{R}) & \xrightarrow{\tilde{\gamma}} & \mathrm{Mob}(\mathbb{RP}^1) \end{array}$$

where π is the natural projection.

Remark B.2. We have defined a real projective structure on M is terms of a differential operator. Equivalently (and more commonly), a real projective structure on M may be defined to be a maximal atlas mapping open sets in M into $\mathbb{RP}^1$ such that the transition functions are restrictions of fractional linear transformations. For the equivalency of the two definitions the reader may consult [**6**].

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